Parameterized Complexity of Two Edge Contraction Problems with Degree Constraints[*]

Rémy Belmonte[1], Petr A. Golovach[1], Pim van 't Hof[1], and Daniël Paulusma[2]

[1] Department of Informatics, University of Bergen, Norway
{remy.belmonte,petr.golovach,pim.vanthof}@ii.uib.no
[2] School of Engineering and Computing Sciences, Durham University, UK
daniel.paulusma@durham.ac.uk

Abstract. Motivated by recent results of Mathieson and Szeider (J. Comput. Syst. Sci. 78(1): 179–191, 2012), we study two graph modification problems where the goal is to obtain a graph whose vertices satisfy certain degree constraints. The REGULAR CONTRACTION problem takes as input a graph G and two integers d and k, and the task is to decide whether G can be modified into a d-regular graph using at most k edge contractions. The BOUNDED DEGREE CONTRACTION problem is defined similarly, but here the objective is to modify G into a graph with maximum degree at most d. We observe that both problems are fixed-parameter tractable when parameterized jointly by k and d. We show that when only k is chosen as the parameter, REGULAR CONTRACTION becomes W[1]-hard, while BOUNDED DEGREE CONTRACTION becomes W[2]-hard even when restricted to split graphs. We also prove both problems to be NP-complete for any fixed $d \geq 2$. On the positive side, we show that the problem of deciding whether a graph can be modified into a cycle using at most k edge contractions, which is equivalent to REGULAR CONTRACTION when $d = 2$, admits an $O(k)$ vertex kernel. This complements recent results stating that the same holds when the target is a path, but that the problem admits no polynomial kernel when the target is a tree, unless NP $\subseteq$ coNP/poly (Heggernes et al., IPEC 2011).

1 Introduction

Graph modification problems play an important role in algorithmic graph theory due to the fact that they naturally appear in numerous practical and theoretical settings. Typically, a graph modification problem takes as input a graph G and an integer k, and the task is to decide whether a graph with certain desirable structural properties can be obtained from G by applying at most k graph operations, such as vertex deletions, edge deletions, edge additions, or a combination of these. The problems VERTEX COVER, FEEDBACK VERTEX SET, MINIMUM FILL-IN and CLUSTER EDITING are just a few famous examples of problems

that fall into this framework. Graph modification problems have received a huge amount of interest in the literature for many decades, and due to the fact that the vast majority of such problems turn out to be NP-hard [11, 15], the area has also been intensively studied from a parameterized complexity point of view.

Moser and Thilikos [14] studied the parameterized complexity of the problem of deciding, given a graph G and an integer k, whether there is a subset of at most k vertices in G whose deletion yields an r-regular graph, where r is a fixed constant. They showed that, for every value of r, this problem is fixed-parameter tractable when parameterized by k, and admits a kernel of size $O(kr(k+r)^2)$. On the other hand, they showed that the problem becomes W[1]-hard for every fixed $r \geq 0$ with respect to the dual parameter $|V(G)| - k$. Mathieson and Szeider [13] showed that the aforementioned positive result by Moser and Thilikos crucially depends on the fact that r is a fixed constant, as they proved the problem to be W[1]-hard when r is given as part of the input. This result by Mathieson and Szeider is a particular case of a much more general result in [13] on graph modification problems involving degree constraints. We refer to [13] for more details, and only mention here that the Classification Theorem in [13] shows that the behavior of the investigated graph modification problems heavily depends on the graph operations that are allowed.

Motivated by the results of Moser and Thilikos [14] and Mathieson and Szeider [13], we study the parameterized complexity of two graph modification problems involving degree constraints when *edge contraction* is the only allowed operation. The parameterized study of graph modification problems with respect to this operation has only recently been initiated, but has already proved to be very fruitful [7–10]. In general, for every graph class $\mathcal{H}$, the $\mathcal{H}$-CONTRACTION problem takes as input a graph G and an integer k, and asks whether there exists a graph $H \in \mathcal{H}$ such that G is *k-contractible* to $\mathcal{H}$, i.e., such that H can be obtained from G by contracting at most k edges. A general result by Asano and Hirata [1] shows that this problem is NP-complete for many natural graph classes $\mathcal{H}$. On the positive side, when parameterized by k, the problem is known to be fixed-parameter tractable when $\mathcal{H}$ is the class of paths or trees [9], bipartite graphs [10, 12], or planar graphs [7]. Interestingly, the problem admits a linear vertex kernel when $\mathcal{H}$ is the class of paths, but does not admit a polynomial kernel when $\mathcal{H}$ is the class of trees, unless NP $\subseteq$ coNP/poly [9].

Before we formally define the two problems studied in this paper and state our results, let us mention one more recent paper that formed a direct motivation for this paper. For any integer $d \geq 1$, let $\mathcal{H}_{\geq d}$ denote the class of graphs with minimum degree at least d. Golovach et al. [8] studied the DEGREE CONTRACTIBILITY problem, which takes as input a graph G and two integers d and k, and asks whether there exists a graph $H \in \mathcal{H}_{\geq d}$ such that G is k-contractible to H. They proved that this problem is fixed-parameter tractable when parameterized jointly by d and k, but becomes W[1]-hard when only k is the parameter. They also showed that the problem is para-NP-complete when parameterized by d by proving the problem to be NP-complete for every fixed value of $d \geq 14$. These results by Golovach et al. [8] raise the question what happens to the com-

plexity of the problem when the objective is not to increase the minimum degree of the input graph, but to decrease the maximum degree instead.

Our Contribution. For any integer $d \geq 1$, let $\mathcal{H}_{\leq d}$ denote the class of graphs that have maximum degree at most d, and let $\mathcal{H}_{=d}$ denote the class of d-regular graphs. In this paper, we study the complexity of different parameterizations of the following two decision problems:

BOUNDED DEGREE CONTRACTION
Instance: A graph G and two integers d and k.
Question: Is there a graph $H \in \mathcal{H}_{\leq d}$ such that G is k-contractible to H?

REGULAR CONTRACTION
Instance: A graph G and two integers d and k.
Question: Is there a graph $H \in \mathcal{H}_{=d}$ such that G is k-contractible to H?

Throughout the paper, we will use n and m to denote the number of vertices and edges, respectively, of the input graph G. Moreover, since edge contractions leave the number of connected components of a graph unchanged, we assume throughout the paper that the input graph G in each of our problems is connected.

In Section 2, we first observe that both problems can be solved in $O((d + k)^{2k} \cdot (n + m))$ time using a simple branching algorithm. This implies that both problems are fixed-parameter tractable when parameterized jointly by d and k, and that both problems are in XP when parameterized by k only. This naturally raises the following two questions:

1. Are the two problems fixed-parameter tractable when parameterized by k?
2. Are the two problems in XP when parameterized by d?

In the remainder of Section 2, we provide strong evidence that the answer to both these questions is "no". We first show that REGULAR CONTRACTION is W[1]-hard when parameterized by k, before proving that BOUNDED DEGREE CONTRACTION is W[2]-hard with the same parameter, even when restricted to the class of split graphs. This implies that neither of the two problems is in FPT, assuming that FPT $\neq$ W[1] and FPT $\neq$ W[2], respectively. The negative answer to question 2, this time under the assumption that P $\neq$ NP, is given in Theorem 3, where we show that both problems are NP-complete for every fixed value of $d \geq 2$, and hence para-NP-complete when parameterized by d. Note that both problems are trivially solvable in polynomial time when $d = 1$. The results of Section 2 are summarized in Table 1.

To complement our hardness results, we show in Section 3 that REGULAR CONTRACTION admits a kernel with at most $6k + 6$ vertices when $d = 2$. Equivalently, we show that the $\mathcal{H}$-CONTRACTION problem admits a linear vertex kernel when $\mathcal{H}$ is the class of cycles. We point out that this problem is para-NP-complete with respect to the dual parameter $|V(G)| - k$, i.e., when parameterized by the length of the obtained cycle, since the problem of deciding whether or not a graph can be contracted to the cycle C_ℓ is NP-complete for every fixed $\ell \geq 4$ [2].

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Parameter	REGULAR CONTRACTION	BOUNDED DEGREE CONTRACTION
d, k	FPT	FPT
k	W[1]-hard	W[2]-hard on split graphs
d	para-NP-complete	para-NP-complete

Table 1. An overview of the results presented in Section 2.

Our kernelization result complements the aforementioned known results stating that $\mathcal{H}$-CONTRACTION admits a linear vertex kernel when $\mathcal{H}$ is the class of paths, but admits no polynomial kernel when $\mathcal{H}$ is the class of trees, unless NP $\subseteq$ coNP/poly [9].

Preliminaries. All graphs considered in this paper are finite, undirected and simple. We refer to the textbook by Diestel [4] for graph terminology and notation not defined below. For a thorough background on parameterized complexity, we refer to the monographs by Downey and Fellows [5].

Let $G = (V, E)$ be a graph and let U be a subset of V. We write $G[U]$ to denote the subgraph of G induced by U. We write $G - U = G[V \setminus U]$, or simply $G - u$ if $U = \{u\}$. We say that two disjoint subsets $U \subseteq V$ and $W \subseteq V$ are *adjacent* if there exist two vertices $u \in U$ and $w \in W$ such that $uw \in E$. The *contraction* of edge uv in G removes u and v from G, and replaces them by a new vertex made adjacent to precisely those vertices that were adjacent to u or v in G. Instead of speaking of the contraction of edge uv, we sometimes say that a vertex u is *contracted onto* v, in which case we use v to denote the new vertex resulting from the contraction. For a set $S \subseteq E$, we write G/S to denote the graph obtained from G by repeatedly contracting an edge from S until no such edge remains. Note that, by definition, edge contractions create neither self-loops nor multiple edges.

Let H be a graph. We say that H is a *contraction* of G if H can be obtained from G by a sequence of edge contractions. We say that G is *k-contractible* to H if H can be obtained from G by at most k edge contractions. An *H-witness structure* $\mathcal{W}$ is a partition of $V(G)$ into $|V(H)|$ nonempty sets $W(x)$, one for each $x \in V(H)$, called *H-witness sets*, such that each $W(x)$ induces a connected subgraph of G, and for all $x, y \in V(H)$ with $x \neq y$, the sets $W(x)$ and $W(y)$ are adjacent in G if and only if x and y are adjacent in H. Clearly, H is a contraction of G if and only if G has an H-witness structure; H can be obtained by contracting each witness set into a single vertex.

2 Contracting to Graphs with Degree Constraints

We start by observing that the problems BOUNDED DEGREE CONTRACTION and REGULAR CONTRACTION are FPT when parameterized jointly by k and d.

Theorem 1. *The problems* BOUNDED DEGREE CONTRACTION *and* REGULAR CONTRACTION *can be solved in time* $O((d + k)^{2k} \cdot (n + m))$.

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Proof. We first present an algorithm for BOUNDED DEGREE CONTRACTION, and then describe how it can be modified to solve REGULAR CONTRACTION in the same running time.

Let (G, d, k) be an instance of BOUNDED DEGREE CONTRACTION. We first check if G has a vertex of degree at least $d + k + 1$. If so, then (G, d, k) is a trivial no-instance, since the contraction of any edge in G decreases the degree of each vertex in G by at most 1. Hence we output "no" in this case. Suppose every vertex in G has degree at most $d + k$, but G has a vertex v such that $d_G(v) \geq d + 1$. In order to contract G to a graph of maximum degree at most d, we must either contract v onto one of its neighbors, or contract all the edges of a path between two of the neighbors of v. In either case, we must contract an edge e incident with a neighbor of v. Since $\Delta(G) \leq d + k$, there are at most $(d + k)^2$ such edges e. We branch on each of them, calling our algorithm recursively for $G' = G/e$ with parameter $k' = k - 1$. Since the parameter decreases by 1 at every step, this branching algorithm runs in time $O((d + k)^{2k} \cdot (n + m))$.

We can also obtain an algorithm for REGULAR CONTRACTION with same running time by replacing the branching rule with the following one: if there is a vertex v with $d_G(v) \neq d$, then we branch over all the edges e that are incident with a vertex in $N_G(v)$. For each branch, we contract the edge e and decrease k by 1. The correctness of this branching rule follows from the observation that if we contract any edge e' that is not incident with a neighbor of v, then the degree of v before and after the contraction is the same. □

We now show that REGULAR CONTRACTION becomes W[1]-hard when only k is chosen as the parameter. In the proof of Theorem 2 below, we will reduce from the following problem:

REGULAR MULTICOLORED CLIQUE
Instance: A regular graph G, an integer k, and a partition $X_1, \ldots, X_k$ of $V(G)$ into k independent sets of size p each.
Question: Does G have a clique $K \subseteq V(G)$ such that $|K \cap X_i| = 1$ for every $i \in \{1, \ldots, k\}$?

It is well-known that the CLIQUE problem, asking whether a given graph has a clique of size k, is W[1]-hard when parameterized by k [5]. Cai [3] proved that this remains true on regular graphs. Using this fact and the standard parameterized reduction from CLIQUE to MULTICOLORED CLIQUE due to Fellows et al. [6], we obtain the following result.

Lemma 1. (★)[3] *The* REGULAR MULTICOLORED CLIQUE *problem is* W[1]-*hard when parameterized by k for d-regular graphs when $k < d < p$.*

We now use the above lemma to prove our first hardness result.

Theorem 2. *The* REGULAR CONTRACTION *problem is* W[1]-*hard when parameterized by k.*

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[3] Proofs marked with a star have been omitted due to page restrictions.

Proof. We reduce from the restricted version of the REGULAR MULTICOLORED CLIQUE problem described in Lemma 1. Let $(G, k, X_1, \ldots, X_k)$ be an instance of this problem where G is a d-regular graph, $p = |X_1| = \ldots = |X_k|$, and $k < d < p$. We construct an instance (G', d', k) of REGULAR CONTRACTION as follows:

– construct a copy of G with the corresponding partition $X_1, \ldots, X_k$ of the vertex set;
– for each $i \in \{1, \ldots, k\}$, construct a vertex x_i and then make the set $X_i \cup \{x_i\}$ into a clique by adding edges;
– make the set $\{x_1, \ldots, x_k\}$ into a clique by adding edges.

Let G' denote the obtained graph, and let $d' = d + p - 1$.

Suppose that G has a clique $K = \{y_1, \ldots, y_k\}$ such that $y_i \in X_i$ for $i \in \{1, \ldots, k\}$. It is straightforward to verify that contracting the edges $x_i y_i$ for $i \in \{1, \ldots, k\}$ in G' results in a d'-regular graph.

Assume now that (G', d', k) is a YES-instance of REGULAR CONTRACTION, i.e., there is a set S of at most k edges such that G'/S is a d'-regular graph. Notice that each x_i in G' has degree $p + k - 1 < p + d - 1 = d'$. Therefore, for each $i \in \{1, \ldots, k\}$, S contains at least one edge incident to x_i. Suppose that S contains an edge $x_i x_j$ for $1 \leq i < j \leq k$. Let G'' be the graph obtained from G' by the contraction of $x_i x_j$, and denote by z the vertex obtained from x_i, x_j. The degree of z in G'' is $2p + k - 2 > p + d + k - 2 = d' + k - 1$. It means that we have to contract at least k edges to obtain a vertex of degree d' from z. It contradicts the assumption that $|S| \leq k$. Hence, for each $i \in \{1, \ldots, k\}$, S contains an edge $x_i y_i$ for $y_i \in X_i$. Since $|S| \leq k$, $S = \{x_1 y_1, \ldots, x_k y_k\}$. We claim that $\{y_1, \ldots, y_k\}$ is a clique in G. To see this, assume that some y_i, y_j are not adjacent in G. Then y_i, y_j are not adjacent in G' but are adjacent in G'/S, and the degree of the vertex obtained from x_i and y_i in G'/S is at least $d + p > d'$. This contradiction to the assumption that G'/S is d'-regular completes the proof of Theorem 2. □

We expect that the arguments in the proof of Theorem 2 can also be used to show that BOUNDED DEGREE CONTRACTION is W[1]-hard when parameterized by k. However, we obtain a stronger result below by proving that BOUNDED DEGREE CONTRACTION is W[2]-hard when parameterized by k, even when restricted to split graphs. This result will be a corollary of the following lemma.

Lemma 2. *The problem of deciding whether the maximum degree of a split graph can be reduced by at least* 1 *using at most k edge contractions is* W[2]-*hard when parameterized by k.*

Proof. We give a reduction from the problem RED-BLUE DOMINATING SET, which takes as input a bipartite graph $G = (R \cup B, E)$ and an integer k, and asks whether there exists a *red-blue dominating set* of size at most k, i.e., a subset $D \subseteq B$ of at most k vertices such that every vertex in R has at least one neighbor in D. This problem, which is equivalent to SET COVER and HITTING SET, is well-known to be W[2]-complete when parameterized by k [5].

Let (G, k) be an instance of RED-BLUE DOMINATING SET, where $G = (R \cup B, E)$ is a bipartite graph with partition classes R and B. We assume that every

vertex in G has degree at least 1. We create a split graph G' from G by making the vertices of R pairwise adjacent, and by adding, for each vertex $u \in R$, $(\Delta(G) - d_G(u)) + k + 2$ new vertices that are made adjacent to u only. Let $B' = V(G') \setminus (R \cup B)$ be the set of all vertices of degree 1 that were added to G this way. Clearly, G' is a split graph, since its vertex set can be partitioned into the clique R and the independent set $B \cup B'$. Observe that each vertex in R has degree $\Delta := \Delta(G') > k + 2$.

We claim that G has a red-blue dominating set of size k if and only if G' can be contracted to a split graph of maximum degree at most $\Delta - 1$ using at most k edge contractions.

First, suppose there is a red-blue dominating set $D \subseteq B$ such that $|D| \le k$. For every $v \in D$, we choose an arbitrary neighbor w of v in R and contract v onto w. Note that contracting v onto w is equivalent to deleting v from the graph due to the fact that $N_{G'}[v] \subseteq N_{G'}[w]$. Since every vertex in R is adjacent to at least one vertex in D, these $|D| \le k$ edge contractions decrease the degree of every vertex in R by at least 1. Since the degree of each vertex in $B \cup B'$ in G' was already at most $\Delta(G) \le \Delta - 1$, the obtained graph has maximum degree at most $\Delta - 1$.

For the reverse direction, suppose there exists a set $S \subseteq E(G')$ of at most k edges such that G'/S has maximum degree at most $\Delta - 1$. We claim that S does not contain any edge whose endpoints both belong to R. To see this, observe that contracting any such edge would create a vertex of degree at least $\Delta + k + 1$, and the degree of such a vertex cannot be decreased to Δ by contracting at most $k - 1$ other edges. Suppose S contains an edge uv such that $u \in R$ and $v \in B'$, and let w be an arbitrary neighbor of u in B. Note that contracting v onto u decreases the degree of u by 1 but leaves the degrees of all other vertices in R unchanged, whereas contracting w onto u decreases the degree of every neighbor of w in R, and of u in particular. Hence we may assume, without loss of generality, that every edge in S is incident with one vertex of R and one vertex of B'. Since the degree of every vertex in R decreases by at least 1 when we contract the edges in S, every vertex in R must be incident with at least one edge in S. This implies that $D := V(S) \cap B$ is a red-blue dominating set of G, where $V(S)$ denotes the set of endpoints of the edges in S. The observation that $|D| \le |S| \le k$ completes the proof. $\qquad\square$

Since an instance (G, k) of the problem defined in Lemma 2 is a yes-instance if and only if $(G, \Delta(G) - 1, k)$ is a yes-instance of Bounded Degree Contraction, we immediately obtain the following result.

Corollary 1. *The* Bounded Degree Contraction *problem is* W[2]-*hard on split graphs when parameterized by* k.

To conclude this section, we also consider the complexity of our two problems when we take only d to be the parameter. The following result shows that both our problems are para-NP-complete with respect to this parameter. Note that both problems can trivially be solved in polynomial time when $d = 1$.

Theorem 3. (★) *The problems* REGULAR CONTRACTION *and* BOUNDED DE-GREE CONTRACTION *are* NP-*complete for any fixed* $d \geq 2$.

3 A Linear Vertex Kernel

In this section, we show that the problem REGULAR CONTRACTION admits a kernel with at most $6k+6$ vertices in case $d = 2$. Since this problem is equivalent to the $\mathcal{H}$-CONTRACTION problem when $\mathcal{H}$ is the class of cycles, we will refer to the problem as CYCLE CONTRACTION throughout this section.

We first introduce some additional terminology. Let G and H be two graphs, and suppose that there exists an H-witness structure $\mathcal{W}$ of G. If a witness set of $\mathcal{W}$ contains more than one vertex of G, then we call it a *big* witness set; a witness set consisting of a single vertex of G is called *small*.

Observation 1 ([9]) *If a graph G is k-contractible to a graph H, then any H-witness structure $\mathcal{W}$ of G satisfies the following three properties:*

- *no witness set of $\mathcal{W}$ contains more than $k + 1$ vertices;*
- *$\mathcal{W}$ has at most k big witness sets;*
- *all the big witness sets of $\mathcal{W}$ together contain at most $2k$ vertices.*

Let G be a graph. A cycle C is *optimal for G* if G can be contracted to C but cannot be contracted to any cycle longer than C. Note that if G is a connected graph that is not a tree, then an optimal cycle for G always exists. The following structural lemma will be used in the correctness proof of our kernelization algorithm.

Lemma 3. *Let (G, k) be a yes-instance of* CYCLE CONTRACTION*, let C be an optimal cycle for G, and let $\mathcal{W}$ be a C-witness structure of G. If G is 2-connected and G contains two vertices u and v such that $d_G(u) = d_G(v) = 2$ and $G - \{u, v\}$ has exactly two connected components G_1 and G_2, then the following three statements hold:*

(i) *either $\{u\}$ and $\{v\}$ are small witness sets of $\mathcal{W}$, or u and v belong to the same big witness set of $\mathcal{W}$;*

(ii) *if u and v belong to the same big witness set $W \in \mathcal{W}$, then W contains all the vertices of G_1 or all the vertices of G_2;*

(iii) *if G_1 and G_2 contain at least $k+1$ vertices each, then $\{u\}$ and $\{v\}$ are small witness sets of $\mathcal{W}$.*

Proof. Suppose G is 2-connected and contains two vertices u and v such that $d_G(u) = d_G(v) = 2$ and $G - \{u, v\}$ has exactly two connected components G_1 and G_2. Let p and q denote the two neighbors of u, and let x and y denote the two neighbors of v. Without loss of generality, suppose $p, x \in V(G_1)$ and $q, y \in V(G_2)$.

To prove statement (i), suppose, for contradiction, that u belongs to a big witness set $W \in \mathcal{W}$ and $v \notin W$. Let $W_1 = (W \setminus \{u\}) \cap V(G_1)$ and $W_2 = $

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$(W \setminus \{u\}) \cap V(G_2)$. Since u has degree 2 in G and $G[W]$ is connected by the definition of a witness set, the graphs $G[W_1]$ and $G[W_2]$ are connected. Moreover, by the definition of G_1 and G_2, there is no edge between W_1 and W_2 in G. Let $\mathcal{W}'$ be the C'-witness structure of G obtained from $\mathcal{W}$ by replacing W with the sets W_1, $\{u\}$, and W_2. Then C' is a cycle that has two more vertices than C. This contradicts the assumption that C is an optimal cycle for G.

We now prove statement (ii). Suppose u and v both belong to the same witness set $W \in \mathcal{W}$. Note that $V(G) \setminus W$ induces a connected subgraph of G, and assume, without loss of generality, that $(V(G) \setminus W) \subseteq V(G_1)$. Then we must have $V(G_2) \subseteq W$.

To prove statement (iii), suppose $|V(G_1)| \geq k+1$ and $|V(G_2)| \geq k+1$. Suppose, for contradiction, that u and v belong to the same big witness set of $\mathcal{W}$. Then W contains all the vertices of either G_1 or G_2 by statement (ii). This implies that W contains at least $k+3$ vertices, contradicting the fact that every big witness set of $\mathcal{W}$ contains at most $k+1$ vertices due to Observation 1. $\quad\square$

We now describe four reduction rules that will be used in our kernelization algorithm for CYCLE CONTRACTION. Each of the reduction rules below takes as input an instance (G, k) of CYCLE CONTRACTION and outputs a reduced instance (G', k') of the same problem, and the rule is said to be *safe* if the two instances (G, k) and (G', k') are either both yes-instances or both no-instances.

Rule 1 *If G is 3-connected and $|V(G)| \geq 2k+4$, then return a trivial no-instance.*

Lemma 4. *Rule 1 is safe.*

Proof. Let G be a 3-connected graph on at least $2k+4$ vertices. We show that G is not k-contractible to a cycle. For contradiction, suppose G is k-contractible to a cycle C. Let $\mathcal{W}$ be a C-witness structure. Then $\mathcal{W}$ has at most three small witness sets, as otherwise for any two small witness sets $\{u\}$ and $\{v\}$ such that u and v are non-adjacent, the graph $G - \{u, v\}$ would be disconnected, contradicting the assumption that G is 3-connected. Since all the big witness sets of $\mathcal{W}$ contain at most $2k$ vertices in total due to Observation 1, this implies that $|V(G)| \leq 2k+3$. This yields the desired contradiction to the assumption that $|V(G)| \geq 2k+4$. $\quad\square$

Rule 2 *If G contains a block B on at least $k+2$ vertices and $V(G) \setminus V(B) \neq \emptyset$, then return a trivial no-instance if $|V(G) \setminus V(B)| \geq k+1$, and return the instance $(G', k - |V(G) \setminus V(B)|)$ otherwise, where G' is the graph obtained from G by exhaustively contracting a vertex of $V(G) \setminus V(B)$ onto one of its neighbors.*

Lemma 5. *Rule 2 is safe.*

Proof. Suppose G contains a block B on at least $k+2$ vertices and $V(G) \setminus V(B) \neq \emptyset$. Then G is not 2-connected and contains at least two blocks.

Suppose (G, k) is a yes-instance, and let $\mathcal{W}$ be C-witness structure of G, where C is a cycle to which G is k-contractible. Since $|V(B)| \geq k+2$, there must

be at least two witness sets of $\mathcal{W}$ that contain vertices of B due to Observation 1. This implies that for every block $B' \neq B$ of G, all the vertices of B' must be contained in one witness set of $\mathcal{W}$, as otherwise there would be two vertex-disjoint paths in G between vertices of B and B'. For the same reason, every witness set of $\mathcal{W}$ contains at least one vertex of B. Consequently, no vertex of $|V(G) \setminus V(B)|$ appears in a small witness set of $\mathcal{W}$.

The above arguments, together with Observation 1, imply that (G, k) is a no-instance if $|V(G) \setminus V(B)| \geq k + 1$. It also implies that the instances (G, k) and $(G', k - |V(G) \setminus V(B)|)$ are equivalent otherwise. $\qquad\square$

Rule 3 *If G contains a block B on at most $k + 1$ vertices and $|V(G) \setminus V(B)| \geq k + 2$, then return the instance $(G', k - |V(B)|)$, where G' is the graph obtained from G by exhaustively contracting a vertex of $V(B)$ onto one of its neighbors.*

Lemma 6. *Rule 3 is safe.*

Proof. Let (G, k) be an instance of CYCLE CONTRACTION, and suppose G has a block B on at most $k + 1$ vertices such that $|V(G) \setminus V(B)| \geq k + 2$. Suppose (G, k) is a yes-instance, and let $\mathcal{W}$ be a C-witness structure of G for some cycle C to which G is k-contractible. Using arguments similar to the ones in the proof of Lemma 5, it can be seen that all the vertices of B must be contained in a single witness set of $\mathcal{W}$, and this witness set contains at least one vertex of $V(G) \setminus V(B)$. This shows that the instances $(G', k - |V(B)|)$ and (G, k) are equivalent. $\qquad\square$

Rule 4 *If G is 2-connected and G contains two vertices u and v such that $d_G(u) = d_G(v) = 2$, the two neighbors p and q of u both have degree 2 in G, and the graph $G - \{u, v\}$ has exactly two connected components that contain at least $k + 2$ vertices each, then return the instance (G', k), where G' is the graph obtained from G by contracting u onto p.*

Lemma 7. *Rule 4 is safe.*

Proof. Let (G, k) be an instance of CYCLE CONTRACTION on which Rule 4 can be applied. Suppose (G, k) is a yes-instance. Let C be an optimal cycle for G, and let $\mathcal{W}$ be a C-witness structure of G. Due to statement (iii) in Lemma 3, $\{u\}$ and $\{v\}$ are small witness sets of $\mathcal{W}$. Then $\mathcal{W}' = \mathcal{W} \setminus \{u\}$ is a C'-witness structure of G', where C' is a cycle containing one less vertex than C. Since the big witness sets of $\mathcal{W}'$ and $\mathcal{W}$ coincide, G' is k-contractible to C'. Hence (G', k) is a yes-instance of CYCLE CONTRACTION.

For the reverse direction, suppose (G', k) is a yes-instance. Let C' be an optimal cycle for G', and let $\mathcal{W}'$ be a C'-witness structure of G'. Consider the vertices p and v in G'. Note that $d_{G'}(p) = d_{G'}(v) = 2$, and that $G' - \{p, v\}$ has exactly two connected components G'_1 and G'_2 that contain at least $k + 1$ vertices each. Hence $\{p\}$ and $\{v\}$ are small witness sets of $\mathcal{W}'$ due to statement (iii) in Lemma 3. For similar reasons, considering the pair (q, v) instead of (p, v), we find that $\{q\}$ is a small witness set of $\mathcal{W}'$. In particular, p and q are in separate small witness sets of $\mathcal{W}'$. Now let $\mathcal{W}$ be the partition of $V(G)$ obtained from $\mathcal{W}'$

by adding the set $\{u\}$. Then $\mathcal{W}$ clearly is a C-witness structure of G, where C is a cycle that has one more vertex than C'. Since the big witness sets of $\mathcal{W}$ and $\mathcal{W}'$ coincide, we conclude that G is k-contractible to C, and hence (G, k) is a yes-instance of CYCLE CONTRACTION. □

Theorem 4. *The* CYCLE CONTRACTION *problem admits a kernel with at most* $6k + 6$ *vertices.*

Proof. We describe a kernelization algorithm for CYCLE CONTRACTION. Given an instance of CYCLE CONTRACTION, the algorithm starts by exhaustively applying the four reduction rules defined above. Let (G, k) be the obtained instance. If G is 3-connected, then $|V(G)| \leq 2k + 3$, as otherwise Rule 1 could be applied. Suppose G is not 2-connected. Since G is connected by assumption, G has at least two blocks. Let B be any block of G. Then $|V(B)| \leq k + 1$, as otherwise Rule 2 could be applied. Moreover, $|V(G) \setminus V(B)| \leq k + 1$ due to the assumption that Rule 3 cannot be applied. Hence $|V(G)| \leq 2k + 2$. Now suppose G is 2-connected. We then apply a final reduction rule: if $|V(G)| \geq 6k + 7$, then return a trivial no-instance. Before showing why this final reduction rule is safe, let us point out that after the application of this final reduction rule, we have obtained an instance (G', k') such that G' has at most $6k + 6$ vertices.

To see why the final reduction rule is safe, suppose, for contradiction, that (G, k) is a yes-instance of CYCLE CONTRACTION such that G is a 2-connected graph on at least $6k + 7$ vertices. Let C be an optimal cycle for G, and let $\mathcal{W}$ be a C-witness structure of G. By Observation 1, at most $2k$ vertices of G belong to big witness sets, which implies that at least $4k + 7$ vertices of G belong to small witness sets. Since $\mathcal{W}$ has at most k big witness sets by Observation 1, there are at most $2k$ vertices in small witness sets that have degree more than 2 in G, namely the ones adjacent to big witness sets. Consequently, there are at least $2k + 7$ vertices in small witness sets that have degree exactly 2, and there must be three small witness sets $\{p\}, \{u\}, \{q\}$ such that $d_G(p) = d_G(u) = d_G(q) = 2$ and p and q are the two neighbors of u. Let $\{v\}$ be a small witness set of $\mathcal{W}$ such that $v \notin \{p, u, q\}$ and $d_G(v) = 2$, and such that the two connected components G_1 and G_2 of the graph $G - \{u, v\}$ contain at least $k + 2$ small witness sets of $\mathcal{W}$ each. Since, apart from the vertices p, u, q and v, there are at least $2k + 3$ other vertices that have degree 2 in G and belong to small witness sets, such a set $\{v\}$ exists. This implies that Rule 4 could have been applied on (G, k), yielding the desired contradiction.

The correctness of our algorithm follows directly from Lemmas 4–7 and from the above proof that the final reduction rule is safe. It remains to argue that our kernelization algorithm runs in polynomial time. It is clear that every reduction rule can be applied in polynomial time. When applying any of the reduction rules, either the number of vertices in the graph or the parameter strictly decreases. This implies that we only apply the reduction rules a polynomial number of times, so the algorithm runs in polynomial time. □

4 Concluding Remarks

We showed that REGULAR CONTRACTION has a linear vertex kernel when $d = 2$. We expect that we can use similar arguments to obtain the same result for BOUNDED DEGREE CONTRACTION when $d = 2$. A more interesting question is whether both problems admit polynomial kernels when $d = 3$.

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