Obtaining Planarity by Contracting Few Edges*

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Abstract. The Planar Contraction problem is to test whether a given graph can be made planar by using at most k edge contractions. This problem is known to be NP-complete. We show that it is fixed-parameter tractable when parameterized by k.

1 Introduction

Numerous problems in algorithmic graph theory, with a large variety of applications in different fields, can be formulated as graph modification problems. A graph modification problem takes as input an n-vertex graph G and an integer k, and the question is whether G can be modified into a graph that belongs to a prescribed graph class, using at most k operations of a certain specified type. Some of the most common graph operations that are used in this setting are vertex deletions, edge deletions and edge additions, leading to famous problems such as Feedback Vertex Set, Odd Cycle Transversal, Minimum Fill-In and Cluster Editing, to name but a few. More recently, the study of graph modification problems allowing only *edge contractions* has been initiated, yielding several results that we will survey below. The contraction of an edge removes both end-vertices of an edge and replaces them by a new vertex, which is made adjacent to precisely those vertices that were adjacent to at least one of the two end-vertices. Choosing edge contraction as the only permitted operation leads to the following decision problem, for each graph class $\mathcal{H}$.

$\mathcal{H}$-Contraction
Instance: A graph G and an integer k.
Question: Does there exist a graph $H \in \mathcal{H}$ such that G can be contracted to H, using at most k edge contractions?

Heggernes et al. [8] presented a $2^{k+o(k)} + n^{O(1)}$ time algorithm for $\mathcal{H}$-Contraction when $\mathcal{H}$ is the class of paths. Moreover, they showed that in this case the problem has a linear vertex kernel. When $\mathcal{H}$ is the class of trees, they showed that the problem can be solved in $4.98^k n^{O(1)}$ time, and that a polynomial kernel does not exist unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$. When the input graph is a chordal graph

* This paper is supported by EPSRC (EP/G043434/1), Royal Society (JP100692), the Research Council of Norway (197548/F20), and ERC (267959).

B. Rovan, V. Sassone, and P. Widmayer (Eds.): MFCS 2012, LNCS 7464, pp. 455–466, 2012.

with n vertices and m edges, then $\mathcal{H}$-Contraction can be solved in $O(n+m)$ time when $\mathcal{H}$ is the class of trees and in $O(nm)$ time when $\mathcal{H}$ is the class of paths [7]. Heggernes et al. [9] proved that $\mathcal{H}$-Contraction is fixed-parameter tractable when $\mathcal{H}$ is the class of bipartite graphs and k is the parameter. This also follows from a more general result from a recent paper by Marx, O'Sullivan, and Razgon [14] on generalized bipartization. Golovach et al. [5] considered the class of graphs of minimum degree at least d for some integer d. They showed that in this case $\mathcal{H}$-Contraction is fixed-parameter tractable when both d and k are parameters, W[1]-hard when k is the parameter, and para-NP-complete when d is the parameter.

The combination of planar graphs and edge contractions has been studied before in a closely related setting. Kamiński, Paulusma and Thilikos [10] showed that for every fixed graph H, there exists a polynomial-time algorithm for deciding whether a given planar graph can be contracted to H. Very recently, this result was improved by Kamiński and Thilikos [11]. They showed that, given a graph H and a planar graph G, the problem of deciding whether G can be contracted to H is fixed-parameter tractable when parameterized by $|V(H)|$.

Our Contribution. We study $\mathcal{H}$-Contraction when $\mathcal{H}$ is the class of planar graphs, and refer to the problem as Planar Contraction. This problem is known to be NP-complete due to a more general result on $\mathcal{H}$-Contraction by Asano and Hirata [2]. We show that the Planar Contraction problem is fixed-parameter tractable when parameterized by k. This result complements the following results on two other graph modification problems related to planar graphs. The problem of deciding whether a given graph can be made planar by using at most k vertex deletions was proved to be fixed-parameter tractable independently by Marx and Schlotter [15], who presented a quadratic-time algorithm for every fixed k, and by Kawarabayashi [12], whose algorithm runs in linear time for every k. Kawarabayashi and Reed [13] showed that deciding whether a graph can be made planar by using at most k edge deletions can also be done in linear time for every fixed k.

Our algorithm for Planar Contraction starts by finding a set S of at most k vertices whose deletion transforms G into a planar graph. Such a set can be found by using either the above-mentioned linear-time algorithm by Kawarabayashi [12] or the quadratic-time algorithm by Marx and Schlotter [15]. The next step of our algorithm is based on the irrelevant vertex technique developed in the graph minors project of Robertson and Seymour [17,19]. We show that if the input graph G has large treewidth, we can find an *edge* whose contraction yields an equivalent, but smaller instance. After repeatedly contracting such irrelevant edges, we invoke Courcelle's Theorem [4] to solve the remaining instance in linear time.

We finish this section by making two remarks that show that we cannot apply the techniques that were used to prove fixed-parameter tractability of the vertex deletion and edge deletion variants of Planar Contraction. First, a crucial observation in the paper of Kawarabayashi and Reed [13] is that any graph that can be made planar by at most k edge deletions must have bounded genus. This property is heavily exploited in the case where the treewidth of the input graph is large. The following example shows that we cannot use this technique in our setting. Take a complete biclique $K_{3,r}$ with partition classes A and B, where $|A| = 3$ and $|B| = r$ for some integer $r \geq 3$. Now make the vertices in A pairwise adjacent and call the resulting graph G_r. Then G_r can be made modified into a planar graph by contracting one of the edges in A. However, the genus of G_r is at least the genus of $K_{3,r}$, which is equal to $\frac{r-2}{2}$ [3].

Second, the problem of deciding whether a graph can be made planar by at most k vertex deletions for some fixed integer k, i.e., k is not part of the input, is called the k-Apex problem. As observed by Kawarabayashi [12] and Marx and Schlotter [15], the class of so-called k-apex graphs (graphs that can be made planar by at most k vertex deletions) is closed under taking minors. This means that the k-Apex problem can be solved in cubic time for any fixed integer k due to deep results by Robertson and Seymour [18]. However, we cannot apply Robertson and Seymour's result on Planar Contraction, because the class of graphs that can be made planar by at most k edge contractions is not closed under taking minors, as the following example shows. Take the complete graph K_5 on 5 vertices. For each edge $e = uv$, add a path P_e from u to v consisting of p new vertices for some integer $p \geq k$. Call the resulting graph G_p^*. Then G_p^* can be made planar by contracting an arbitrary edge of the original K_5. However, if we remove all edges of this K_5, we obtain a minor of G_p^* that is a subdivision of the graph K_5. In order to make this minor planar, we must contract all edges of a path P_e, so we need at least $p + 1 > k$ edge contractions.

2 Preliminaries

Throughout the paper we consider undirected finite graphs that have no loops and no multiple edges. Whenever we consider a graph problem, we use n to denote the number of vertices of the input graph. Let $G = (V, E)$ be a graph and let S be a subset of V. We write $G[S]$ to denote the subgraph of G *induced* by S, i.e., the subgraph of G with vertex set S and edge set $\{uv \mid u, v \in S \text{ with } uv \in E\}$. We write $G - S = G[V \setminus S]$, and for any subgraph H of G, we write $G - H$ to denote $G - V(H)$. We say that two disjoint subsets $U \subseteq V$ and $W \subseteq V$ are *adjacent* if there exist two vertices $u \in U$ and $w \in W$ such that $uw \in E$. Let H be a graph that is not necessarily vertex-disjoint from G. Then $G \cup H$ denotes the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, and $G \cap H$ denotes the graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$.

The *contraction* of edge uv in G removes u and v from G, and replaces them by a new vertex made adjacent to precisely those vertices that were adjacent to u or v in G. A graph H is a *contraction* of G if H can be obtained from G by a sequence of edge contractions. Alternatively, we can define a contraction of G as follows. An *H-witness structure* $\mathcal{W}$ is a partition of $V(G)$ into $|V(H)|$ nonempty sets $W(x)$, one for each $x \in V(H)$, called *H-witness sets*, such that each $W(x)$ induces a connected subgraph of G, and for all $x, y \in V(H)$ with $x \neq y$, the sets $W(x)$ and $W(y)$ are adjacent in G if and only if x and y are adjacent in H.

Clearly, H is a contraction of G if and only if G has an H-witness structure; H can be obtained by contracting each witness set into a single vertex. A *witness edge* is an edge of G whose end-vertices belong to two different witness sets.

Let $\mathcal{W}$ be an H-witness structure of G. For our purposes, we sometimes have to contract edges in G such that the resulting graph does *not* contain H as contraction. In order to do this it is necessary to *destroy* $\mathcal{W}$, i.e., to contract at least one witness edge in $\mathcal{W}$. After all, every edge that is not a witness edge has both its end-vertices in the same witness set of $\mathcal{W}$, which means that contracting such an edge yields an H-witness structure of a contraction of G. Hence, contracting all such non-witness edges transforms G into H itself. Note that if we destroy $\mathcal{W}$ by contracting a witness edge e in $\mathcal{W}$, the obtained graph still has H as a contraction if e was a non-witness edge in some other H-witness structure of G. Hence, in order to obtain a graph that does not have H as a contraction, it is necessary and sufficient to destroy *all* H-witness structures of G.

A *planar graph* G is a graph that can be embedded in the plane, i.e., that can be drawn in the plane so that its edges intersect only at their end-vertices. A graph that is actually drawn in such a way is called a *plane graph*, or an *embedding* of the corresponding planar graph. A plane graph G partitions the rest of the plane into a number of connected regions, called the *faces* of G. Each plane graph has exactly one unbounded face, called the *outer* face; all other faces are called *inner faces*. Let C be a cycle in a plane graph G. Then, by the Jordan Curve Theorem, C divides the plane in exactly two regions: the *outer* region of C, containing the outer face of G, and the *inner* region of C. We say that a vertex u of G lies *inside* C if u is in the inner region of C. Similarly, u lies *outside* C if u is in the outer region of C. The *interior* of C with respect to G, denoted $\text{interior}_G(C)$, is the set of all vertices of G that lie inside C. We also call these vertices *interior* vertices of C. We say that C *separates* the vertices that lie inside C from the vertices that lie outside C. A sequence of mutually vertex-disjoint cycles $C_1, \ldots, C_q$ in a plane graph is called *nested* if there exist disks $\Delta_1, \ldots, \Delta_q$ such that C_i is the boundary of Δ_i for $i = 1, \ldots, q$, and $\Delta_{i+1} \subset \Delta_i$ for $i = 1, \ldots, q-1$. We also refer to such a sequence of nested cycles as *layers*. We say that a vertex u lies *between* two nested cycles C_i and C_j with $i < j$ if u lies in the inner region of C_i and in the outer region of C_j.

A graph G contains a graph H as a *minor* if G can be modified to H by a sequence of edge contractions, edge deletions and vertex deletions. Note that a graph G contains a graph H as a minor if and only if G contains a subgraph that contains H as a contraction. The *subdivision* of an edge $e = uv$ in a graph G removes e from G and replaces it by a new vertex w that is made adjacent to u and v. A *subdivision* of a graph G is a graph obtained from G after performing a sequence of edge subdivisions. In Figure 1, three examples of an *elementary wall* are given. The unique cycle that forms the boundary of the outer face is called the *perimeter* of the wall. A *wall* W of height h is a subdivision of an elementary wall of height h and is well-known to have a unique planar embedding. We also call the facial cycle of W corresponding to the perimeter of the original elementary wall the *perimeter* of W, and we denote this cycle by $P(W)$.

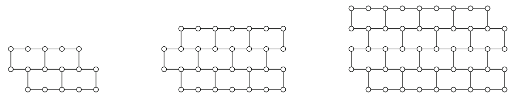

Fig. 1. Elementary walls of height 2, 3, and 4 with perimeters of length 14, 22, and 30, respectively

The $r \times r$ *grid* has all pairs (i, j) for $i, j = 0, 1, \ldots, r-1$ as the vertex set, and two vertices (i, j) and (i', j') are joined by an edge if and only if $|i-i'|+|j-j'| = 1$. The *side length* of an $r \times r$ grid is r. A well-known result of Robertson and Seymour [16] states that, for every integer r, any planar graph with no $r \times r$ grid minor has treewidth at most $6r-5$. Although it is not known whether a largest grid minor of a planar graph can be computed in polynomial time, there exist several constant-factor approximation algorithms. In our algorithm we will use one by Gu and Tamaki [6]. For any graph G, let $\mathrm{gm}(G)$ be the largest integer r such that G has an $r \times r$ grid as a minor. Gu and Tamaki [6] showed that for every constant $\epsilon > 0$, there exists a constant $c_\epsilon > 3$ such that an $r \times r$ grid minor in a planar graph G can be constructed in time $O(n^{1+\epsilon})$, where $r \geq \mathrm{gm}(G)/c_\epsilon$. Because we can obtain a wall of height $\lfloor r/2 \rfloor$ as a subgraph from an $r \times r$ grid minor by deleting edges and vertices, their result implies the following theorem.

Theorem 1 ([6]). *Let G be a planar graph, and let h^* be the height of a largest wall that appears as a subgraph in G. For every constant $\epsilon > 0$, there exists a constant $c_\epsilon > 3$ such that a wall in G with height at least h^*/c_ϵ can be constructed in time $O(n^{1+\epsilon})$.*

3 Fixed-Parameter Tractability of Planar Contraction

For our algorithm we need the aforementioned result of Kawarabayashi [12].

Theorem 2 ([12]). *For every fixed integer k, it is possible to find in $O(n)$ time a set S of at most k vertices in an n-vertex graph G such that $G - S$ is planar, or conclude that such a set S does not exist.*

We also need three lemmas, whose proofs are left out due to space restrictions.

Lemma 1. *If a graph $G = (V, E)$ can be contracted to a planar graph by using at most k edge contractions, then there exists a set $S \subseteq V$ with $|S| \leq k$ such that $G - S$ is planar.*[1]

When k is fixed, we write k-Planar Contraction instead of Planar Contraction. A seminal result of Courcelle [4] states that on any class of graphs of bounded treewidth, every problem expressible in monadic second-order logic can be solved in time linear in the number of vertices of the graph.

[1] As an aside, we point out that the reverse of this statement is not true. For instance, take a K_5 and subdivide each of its edges $p \geq 3$ times. The resulting graph can be made planar by one vertex deletion, but at least $p-1$ edge contractions are required.

Lemma 2. *For every fixed integer k, the k-PLANAR CONTRACTION problem can be expressed in monadic second-order logic.*

Lemma 3. *Let B be a planar graph that has an embedding with two nested cycles C_1 and C_2, such that C_1 is the boundary of its outer face and C_2 is the boundary of an inner face, and such that there are at least two vertex-disjoint paths that join vertices of C_1 and C_2. Let I be a graph with $B \cap I = C_2$ such that $R = B \cup I$ is planar. Then R has an embedding such that C_1 is the boundary of the outer face.*

We are now ready to present our main theorem, which shows that PLANAR CONTRACTION is fixed-parameter tractable when parameterized by k. At some places in our proof of Theorem 3 we allow constant factors (independent of k) to be less than optimal in order to make the arguments easier to follow.

Theorem 3. *For every fixed integer k and every constant $\epsilon > 0$, the k-PLANAR CONTRACTION problem can be solved in $O(n^{2+\epsilon})$ time.*

Proof. Let G be a graph on n vertices, and let k be some fixed integer. If G has connected components $L_1, \ldots, L_q$ for some $q \geq 2$, then we solve for every possible tuple $(k_1, \ldots, k_q)$ with $\sum_{i=1}^{q} k_i = k$ the instances $(L_1, k_1), \ldots, (L_q, k_q)$. Hence, we may assume without loss of generality that G is connected. We apply Theorem 2 to decide in $O(n)$ time whether G contains a subset S of at most k vertices such that $G - S$ is planar. If not, then we return `no` due to Lemma 1. Hence, from now on we assume that we have found such a set S. We write $H = G - S$.

Choose $\epsilon > 0$. We apply Theorem 1 on the graph H to find in $O(n^{1+\epsilon})$ time a subgraph W of H that is a wall with height $h \geq h^*/c_\epsilon$, where h^* denotes the height of a largest wall in H and $c_\epsilon > 3$ is some constant.

Suppose that $h \leq \lceil\sqrt{2k+1}\rceil(12k+10)$. Then $h^* \leq c_\epsilon h \leq c_\epsilon \lceil\sqrt{2k+1}\rceil(12k+10)$, i.e., the height of a largest wall in H is bounded by a constant. Consequently, the treewidth of H is bounded by a constant [16]. Since deleting a vertex from a graph decreases the treewidth by at most 1, the treewidth of G is at most $|S| \leq k$ larger than the treewidth of H. Because k is fixed, this means that the treewidth of G is bounded by a constant as well. Then Lemma 2 tells us that we may apply Courcelle's Theorem [4] to check in $O(n)$ time if G can be modified into a planar graph by using at most k edge contractions.

Now suppose that $h > \lceil\sqrt{2k+1}\rceil(12k+10)$. We consider some fixed planar embedding of H. For convenience, whenever we mention the graph H below, we always refer to this fixed embedding. The wall W is contained in some connected component $\tilde{H}$ of H, and we assume without loss of generality that all other connected components of H lie outside $P(W)$. Inside $P(W)$, we choose $2k+1$ mutually vertex-disjoint subwalls $W_1, \ldots, W_{2k+1}$ of height $12k+8$ that are packed inside W in $\lceil\sqrt{2k+1}\rceil$ rows of $\lceil\sqrt{2k+1}\rceil$ subwalls, such that vertices of distinct subwalls are not adjacent; see Figure 2. Inside each W_i, we choose a subwall W_i' of height $12k+6$ such that the perimeters of W_i and W_i' are vertex-disjoint; see Figure 2 for a depiction of W_i and W_i' in case $k = 0$. By definition, the

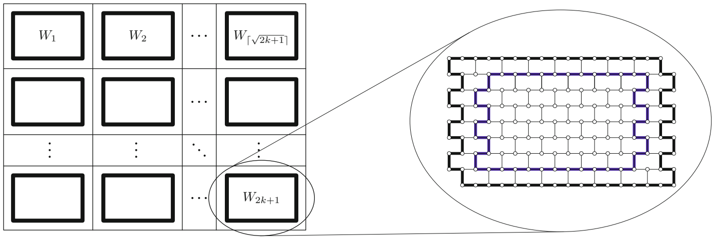

Fig. 2. On the left, a schematic depiction of the wall W with height $h > \lceil\sqrt{2k+1}\rceil(12k+10)$ and the way the subwalls $W_1, \ldots, W_{2k+1}$, each with height $12k+8$, are packed within W. On the right, a more detailed picture of a subwall W_i of height 8 in case $k = 0$. The bold blue edges indicate the perimeter of the smaller subwall W_i' of height 6.

inner region of $P(W_i)$ is the region that contains the vertices of W_i', and the inner region of $P(W_i')$ is the region that contains no vertex of $P(W_i)$. Note that the interiors of $P(W_i)$ and $P(W_i')$ are defined with respect to (the fixed planar embedding of) the graph H. Hence, these interiors may contain vertices of H that do not belong to W, as W is a subgraph of H.

We now consider the graph G. Recall that $H = G - S$, and that G is not necessarily planar. Hence, whenever we speak about the interior of some cycle below, we always refer to the interior of that cycle with respect to the fixed planar embedding of H. For $i = 1, \ldots, 2k + 1$, let $S_i \subseteq S$ be the subset of vertices of S that are adjacent to an interior vertex of $P(W_i')$. Observe that the sets S_i are not necessarily disjoint, since a vertex of S might be adjacent to interior vertices of $P(W_i')$ for several values of i. Also note that no vertex of S belongs to W, since W is a wall in the graph $H = G - S$. We can construct the sets S_i in $O(n)$ time, because the number of edges of G is $O(n)$. The latter can be seen as follows. The number of edges in G is equal to the sum of the number of edges of H, the number of edges between H and S, and the number of edges of $G[S]$. Because H is planar, the number of edges of H is at most $5|V(H)| \leq 5n$. Hence, the number of edges of G is at most $5n + kn + \frac{1}{2}k(k-1) = O(n)$ for fixed k.

We say that a set S_i is of *type 1* if S_i is non-empty and if every vertex $y \in S_i$ also belongs to some set S_j for $j \neq i$, i.e., every vertex $y \in S_i$ is adjacent to some vertex z that lies inside $P(W_j')$ for some $j \neq i$; see Figure 3 for an illustration. Otherwise we say that S_i is of *type 2*. We can check in $O(n)$ time how many sets S_i are of type 1. We claim the following.

Claim 1. If there are at least $k+1$ sets S_i of type 1, then (G, k) is a no-instance.

We prove Claim 1 as follows. Suppose that there exist $\ell \geq k + 1$ sets S_i of type 1, say these sets are $S_1, \ldots, S_\ell$. Then for each $i = 1, \ldots, \ell$ we can define a K_5-witness structure $\mathcal{X}_i$ of a subgraph of G as follows. We divide the perimeter of W_i into three connected non-empty parts in the way illustrated in Figure 3. The vertices of each part will form a separate witness set of $\mathcal{X}_i$; let us call these witness sets X_i^1, X_i^2, X_i^3. Let H_i' be the subgraph of H induced by the vertices

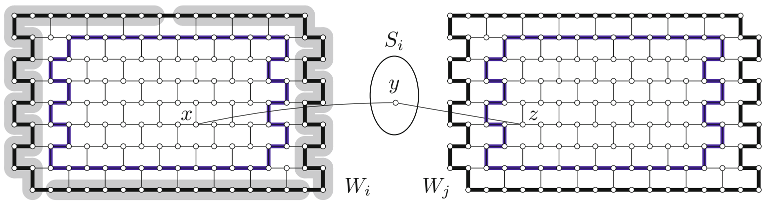

Fig. 3. Two subwalls W_i and W_j, where the bold blue edges indicate the perimeters $P(W_i')$ and $P(W_j')$ of the smaller subwalls W_i' and W_j'. Vertex y is in S_i, since it is adjacent to an interior vertex x of $P(W_i')$. If, for every $y \in S_i$, there is an edge between y and an interior vertex z of $P(W_j')$ for some $j \neq i$, then S_i is of type 1. The three shaded areas indicate how the perimeter of W_i is divided into three non-empty parts, each forming a separate witness set of a K_5-witness structure $\mathcal{X}_i$ of a subgraph of G.

that lie inside $P(W_i)$ in H. The fourth set X_i^4 of the witness structure $\mathcal{X}_i$ consists of all the vertices of the connected component of H_i' that contains W_i'. Let G_i' be the graph obtained from G by deleting all the vertices of $P(W_i)$ and all the vertices that lie inside $P(W_i)$ in H, i.e., $G_i' = G - P(W_i) - \text{interior}_H(P(W_i))$. Let D be the connected component of G_i' that contains the perimeter $P(W)$ of the large wall W. It is clear that D contains all vertices that are on or inside $P(W_j)$ for every $j \neq i$. Hence, due to the assumption that S_i is of type 1, all vertices of S_i also belong to D. The fifth set X_i^5 is defined to be the vertex set of D. Let us argue why these five sets form a K_5-witness structure of a subgraph of G.

It is clear that each of the sets X_i^1, X_i^2, X_i^3 is connected, and that they are pairwise adjacent. The set X_i^4 is connected by definition. The choice of the subwall W_i' within W_i ensures that X_i^4 is adjacent to each of the sets X_i^1, X_i^2, X_i^3. Let us consider the set X_i^5. By definition, X_i^5 is connected. Since X_i^5 contains the perimeter $P(W)$ of the large wall W and W_i lies inside $P(W)$, set X_i^5 is adjacent to X_i^1, X_i^2 and X_i^3. Since S_i is of type 1 and hence non-empty, there is a vertex $y \in S_i$ that is adjacent to a vertex x that lies inside $P(W_i')$ by the definition of S_i. We already argued that X_i^5 contains all vertices of S_i, so $y \in X_i^5$. Recall that $\tilde{H}$ is the unique connected component of H that contains the wall W, and that all connected components of H other than $\tilde{H}$ were assumed to lie outside $P(W)$ in H. Because x lies inside $P(W_i')$, this means that x is in the connected component of H_i' that contains W_i', implying that $x \in X_i^4$. Consequently, the edge between x and y ensures the adjacency between X_i^4 and X_i^5.

We now consider the ℓ different K_5-witness structures $\mathcal{X}_i$ of subgraphs of G defined in the way described above, one for each $i \in \{1, \ldots, \ell\}$. Let us see how such a K_5-witness structure $\mathcal{X}_i$ can be destroyed by using edge contractions only. Denote by E_i the set of edges of G incident with the vertices of $X_i^1 \cup \cdots \cup X_i^4$ for $i = 1, \ldots, \ell$. We can only destroy a witness structure $\mathcal{X}_i$ by edge contractions if we contract the edges of at least one path that has its endvertices in different witness sets of $\mathcal{X}_i$ and its inner vertices (in case these exist) not belonging to any witness set of $\mathcal{X}_i$. Clearly, such a path always contains an edge of E_i. Hence, in order to destroy $\mathcal{X}_i$, we have to contract at least one edge of E_i. Because the sets $E_1, \ldots, E_\ell$ are pairwise disjoint by the construction of the witness structures $\mathcal{X}_i$, we must use at least $\ell \geq k+1$ edge contractions in order to make G planar. Hence, G is a no-instance. This proves Claim 1.

Due to Claim 1, we are done if there are at least $k+1$ sets S_i of type 1. Note that every step in our algorithm so far took $O(n^{1+\epsilon})$ time, as desired. Suppose that we found at most k sets S_i of type 1. Because the total number of sets S_i is $2k+1$, this means that there are at least $k+1$ sets S_i of type 2. Let S_i be a set of type 2. If S_i is non-empty, then S_i contains a vertex x that is not adjacent to an interior vertex of $P(W'_j)$ for any $j \neq i$, as otherwise S_i would be of type 1. Consequently, S_i is the only set of type 2 that contains x. Since $|S| \leq k$ and there are at least $k+1$ sets of type 2, at least one of them must be empty. Without loss of generality, we assume from now on that $S_1 = \emptyset$.

We will now exploit the property that $S_1 = \emptyset$, i.e., that none of the vertices in the interior of $P(W'_1)$ is adjacent to any vertex of S. We define a *triple layer* as the perimeter of a wall with the perimeters of its two largest proper subwalls inside, such that the three perimeters are mutually vertex-disjoint, and the *middle* perimeter is adjacent to the *outer* and *inner* perimeter. We define a sequence of nested triple layers in the same way as we defined a sequence of layers in Section 2. Because W'_1 has height $12k+6$, there exist two adjacent vertices u and v inside $P(W'_1)$, such that u and v are separated from the vertices outside $P(W'_1)$ by $2k+1$ nested triple layers $L_1, \ldots, L_{2k+1}$, i.e., u and v lie inside the inner perimeter of triple layer L_{2k+1}.

Let G' be the graph obtained from G after contracting uv. The following claim shows that uv is an "irrelevant" edge, i.e., that uv may be contracted without loss of generality.

Claim 2. (G,k) is a yes-instance if and only if (G',k) is a yes-instance.

We prove Claim 2 as follows. First suppose that (G,k) is a yes-instance. This means that G can be modified into a planar graph F by at most k edge contractions. Let $E' \subseteq E(G)$ be a set of at most k edges whose contraction modifies G into F. Observe that we can contract the edges in E' in any order to obtain F from G. If $uv \in E'$, then we can first contract uv to obtain the graph G', and then contract the other edges in E' to modify G' into the planar graph F. If $uv \notin E'$, then we first contract the edges in E' to modify G into F, and then contract the edge uv. This leads to a graph F'. Since planar graphs are closed under edge contractions, F' is planar. Moreover, F' can also be obtained from G' by contracting the edges in E'. We conclude that (G',k) is a yes-instance.

Now suppose that (G',k) is a yes-instance. This means that G' can be modified into a planar graph F' by at most k edge contractions. Let $E' \subseteq E(G')$ be a set of at most k edges whose contraction modifies G' into F'. Let F be the graph obtained from G by contracting all the edges of E'. We will show that F is planar as well.

Recall that $S_1 = \emptyset$, and that we defined $2k+1$ triple layers $L_1, \ldots, L_{2k+1}$ inside $P(W'_1)$. Let Q_i, Q'_i, and Q''_i denote the three perimeters in H that form the triple layer L_i for $i = 1, \ldots, 2k+1$, where Q_i is the outer perimeter, Q'_i the middle perimeter, and Q''_i the inner perimeter. Let Y_i be the set of all vertices of H that are in $Q_i \cup Q'_i \cup Q''_i$ or that lie in the intersection of the inner region of Q_i and the outer region of Q''_i, i.e., Y_i is the set of vertices that lie on or "in between" the perimeters Q_i and Q''_i in H. Because we applied at most k edge contractions in G', there exists a set Y_i, for some $1 \leq i \leq 2k+1$, such that none of its vertices is incident with an edge in E'. This means that L_i is a triple layer in F' as well. We consider a planar embedding of F', in which Q''_i is in the inner region of Q'_i, and Q'_i is in the inner region of Q_i; for convenience, we will denote this planar embedding by F' as well.

We will now explain how to apply Lemma 3. We define C_1 and C_2 to be the perimeters Q'_i and Q''_i, respectively. We define B as the subgraph of F' induced by the vertices that either are in $Q'_i \cup Q''_i$ or lie between Q'_i and Q''_i in F'. Here, we assume that B is connected, as we can always place connected components of B that do not contain vertices from $Q'_i \cup Q''_i$ outside Q'_i. Because Q'_i and Q''_i are perimeters of subwalls in H, and Q''_i is contained inside Q'_i, there exist at least two vertex-disjoint paths P_1, P_2 in H joining Q'_i and Q''_i using vertices of Y_i only. Because none of the vertices in Y_i is incident with an edge in E', the two paths P_1, P_2 are also vertex-disjoint in F', and consequently in B.

We now construct the graph I. Because F' is a contraction of G', and G' is a contraction of G, we find that F' is a contraction of G. Let $\mathcal{W}$ be an F'-witness structure of G corresponding to contracting exactly the edges of $E' \cup \{uv\}$ in G. Then we define I to be the subgraph of G induced by the union of the vertices of all the witness sets $W(x)$ with x on or inside Q''_i in F'. Just as we may assume that B is connected, we may also assume that the subgraph of F' induced by the vertices that lie on or inside Q''_i is connected. Because witness sets are connected by definition, we then find that I is connected.

Because the edge uv is contracted when G is transformed into F', u and v belong to the same witness set of $\mathcal{W}$. Let x^* be the vertex of F', such that u and v are in the witness set $W(x^*)$. Recall that all the vertices of Q''_i and the vertices u and v belong to the wall W'_1. Since u and v lie inside Q''_i in H and walls have a unique plane embedding, x^* lies inside Q''_i in F'. Hence, u and v are vertices of I. Also recall that none of the vertices of Y_i, and none of the vertices of Q''_i in particular, is incident with an edge of E'. Hence, the vertices of Q''_i correspond to witness sets of $\mathcal{W}$ that are singletons, i.e., that have cardinality 1. This means that we can identify each vertex of Q''_i in F' with the unique vertex of G in the corresponding witness set. Hence, we obtain that $B \cap I = Q''_i = C_2$.

We now prove that $R = B \cup I$ is planar. For doing this, we first prove that B contains no vertex x with $W(x) \cap S \neq \emptyset$, and that I contains no vertex from S. To see that B contains no vertex x with $W(x) \cap S \neq \emptyset$, assume that x is a vertex of B and s is a vertex of S with $s \in W(x)$. Recall that no vertex from Q'_i is incident with an edge in E'. Hence, we can identify each vertex in Q'_i in F' with the unique vertex of the corresponding witness set, just as we did earlier with the vertices of Q''_i. Because $s \in W(x)$, this means that x is not in Q'_i. Because B is connected, we find that F' contains a path from x to a vertex y in Q'_i that contains no vertex from Q_i. Note that since y is in Q_i', y is a vertex in G as well. Because $W(x)$ induces a connected subgraph of G by definition, this path can be transformed into a path in G from s to y that does not contain a vertex from Q_i. This is not possible, because $S_1 = \emptyset$ implies that every path in G from s to y must go through Q_i.

We now show that I contains no vertex from S. In order to obtain a contradiction, assume that I contains a vertex $s \in S$. Because I is connected, this means that G contains a path from s to a vertex in Q_i'' that contains no vertex from Q_i (and also no vertex from Q_i'). This is not possible, because $S_1 = \emptyset$.

Let R' be the subgraph of G induced by the vertices in the sets $W(x)$ with $x \in V(B)$ and the vertices of I. Since we proved that R' contains no vertices from S, R' is a subgraph of H. Consequently, R' is planar because H is planar. As a result, R is planar because R can be obtained from R' by contracting all edges in every set $W(x)$ with $x \in V(B)$, and planar graphs are closed under edge contractions. As we have shown that $R = B \cup I$ is planar, we are now ready to apply Lemma 3. This lemma tells us that R has an embedding $\mathcal{R}$, such that $Q_i' = C_1$ is the boundary of the outer face. Now consider the embedding that we obtain from the (plane) graph F' by removing all vertices that lie inside Q_i'. We combine this embedding with $\mathcal{R}$ to obtain a plane embedding of a graph F^*. We can obtain F from F^* by contracting all edges in E' that are incident to a vertex in I; recall that u and v are both in I and that uv is not an edge of E'. Because planar graphs are closed under edge contractions, this means that F is planar. This completes the proof of Claim 2.

We can find the irrelevant edge uv mentioned just above Claim 2 in $O(n)$ time. Since all other steps took $O(n^{1+\epsilon})$ time, we used $O(n^{1+\epsilon})$ time so far. After finding the edge uv, we contract it and continue with the smaller graph G'. Because removing S will make G' planar as well, we can keep S instead of applying Theorem 2 again. Hence, we apply Theorem 2 only once. Because G has n vertices, and every iteration reduces the number of vertices by exactly one, the total running time of our algorithm is $O(n^{2+\epsilon})$. This completes the proof. □

4 Conclusions

We proved that Planar Contraction is fixed-parameter tractable when parameterized by k. Very recently, Abello et al. [1] independently showed that the closely related problem that is to test whether a given graph can be made planar by contracting the edges of at most k mutually vertex-disjoint subgraphs, each of which of size at most ℓ, can be solved in quadratic time for any fixed k and $\ell \geq 2$. Their algorithm can easily be modified to show that k-Planar Contraction can be solved in quadratic time for any fixed k (just as we can modify our algorithm to solve their problem).

A natural direction for future work is to consider the class $\mathcal{H}$ that consists of all H-minor free graphs for some graph H and to determine the parameterized complexity of $\mathcal{H}$-Contraction for such graph classes. Our proof techniques rely on the fact that we must contract to a planar graph, and as such they cannot be used directly for this variant. Hence, we pose this problem as an open problem.

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