Exploring Subexponential Parameterized Complexity of Completion Problems*

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Abstract

Let $\mathcal{F}$ be a family of graphs. In the $\mathcal{F}$-Completion problem, we are given an n-vertex graph G and an integer k as input, and asked whether at most k edges can be added to G so that the resulting graph does not contain a graph from $\mathcal{F}$ as an induced subgraph. It appeared recently that special cases of $\mathcal{F}$-Completion, the problem of completing into a chordal graph known as Minimum Fill-in, corresponding to the case of $\mathcal{F}$ being the set of all cycles of length more than 3, and the problem of completing into a split graph, i.e., the case of $\mathcal{F} = \{C_4, 2K_2, C_5\}$ (the cycle on four vertices, its complement, and the cycle on five vertices), are solvable in parameterized subexponential time $2^{O(\sqrt{k}\log k)}n^{O(1)}$.

In this paper we prove that completions into several well studied classes of graphs without short cycles also admit parameterized subexponential time algorithms by showing that:

- For $\mathcal{F} = \{C_4, P_4\}$, i.e., a cycle and a path on four vertices, $\mathcal{F}$-Completion, also known as Trivially Perfect Completion, is solvable in parameterized subexponential time $2^{O(\sqrt{k}\log k)}n^{O(1)}$.
- The problems known in the literature as Pseudosplit Completion, the case where $\mathcal{F} = \{2K_2, C_4\}$, and Threshold Completion, where $\mathcal{F} = \{2K_2, P_4, C_4\}$, are solvable in time $2^{O(\sqrt{k}\log k)}n^{O(1)}$.

We complement our algorithms for $\mathcal{F}$-Completion with the following lower bounds:

- For $\mathcal{F} = \{2K_2\}$, $\mathcal{F} = \{C_4\}$, $\mathcal{F} = \{P_4\}$, and $\mathcal{F} = \{2K_2, P_4\}$, $\mathcal{F}$-Completion cannot be solved in time $2^{o(k)}n^{O(1)}$ unless the Exponential Time Hypothesis (ETH) fails.

Our upper and lower bounds provide a complete picture of subexponential parameterized complexity of $\mathcal{F}$-Completion problems for $\mathcal{F} \subseteq \{2K_2, C_4, P_4\}$.

1 Introduction

Let $\mathcal{F}$ be a family of graphs. In this paper we study the following generic $\mathcal{F}$-Completion problem.

$\mathcal{F}$-Completion	**Parameter:** k
Input: A graph $G = (V, E)$ and a non-negative integer k.	
Question: Does there exist a supergraph $H = (V, E \cup S)$ of G, such that $\lvert S\rvert \leq k$ and H contains no graph from $\mathcal{F}$ as an induced subgraph?	

The $\mathcal{F}$-Completion problem is a subclass of graph modification problems where one is asked to apply a bounded number of changes to an input graph to obtain a graph with some

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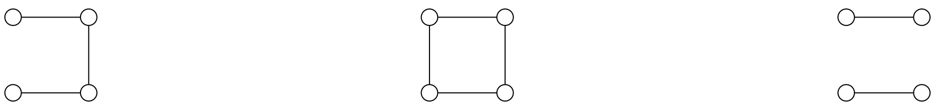
property. Graph modification problems arise naturally in many branches of sciences and have been studied extensively during the past 40 years, see e.g. [10, pages 190 – 205]. Interestingly enough, despite the long study of the problem, there is no known dichotomy classification of $\mathcal{F}$-COMPLETION explaining for which classes $\mathcal{F}$ the problem is solvable in polynomial time and for which the problem is NP-complete [25].

One of the motivations to study completion problems in graph algorithms comes from their intimate connections to different width parameters. For example, the treewidth of a graph, one of the most fundamental graph parameters, is the minimum over all possible completions into a chordal graph of the maximum clique size minus one [2]. The treedepth of a graph, also known as the vertex ranking number, the ordered chromatic number, and the minimum elimination tree height, plays a crucial role in the theory of sparse graphs developed by Nešetřil and Ossona de Mendez [22]. Mirroring the connection between treewidth and chordal graphs, the treedepth of a graph can be defined as the largest clique size in a completion to a *trivially perfect graph*. Similarly, the vertex cover of a graph can be defined as the minimum taken over all completions to a *threshold graph* of the largest clique size minus one.



(a) P_4 (b) C_4 (c) $2K_2 = \overline{C_4}$.

Figure 1: Forbidden subgraphs. *Trivially perfect graphs* are $\{C_4, P_4\}$-free, threshold graphs are $\{2K_2, P_4, C_4\}$-free, and cographs are P_4-free.

Parameterized algorithms for completion problems. For a long time in parameterized complexity the main focus of studies in $\mathcal{F}$-COMPLETION was for the case when $\mathcal{F}$ was an infinite family of graphs, e.g., MINIMUM FILL-IN or INTERVAL COMPLETION [16, 21, 23]. (The problem MINIMUM FILL-IN is also known as MINIMUM TRIANGULATION and CHORDAL COMPLETION, the problem of adding edges to obtain a chordal graph.) This was mainly due to the fact that when $\mathcal{F}$ is a finite family, $\mathcal{F}$-COMPLETION is solvable on an n-vertex graph in time $f(k) \cdot n^{\mathcal{O}(1)}$ for some function f by a simple branching argument; this was first observed by Cai [4]. More precisely, if the maximum number of non-edges in a graph from $\mathcal{F}$ is d, then the corresponding $\mathcal{F}$-COMPLETION is solvable in time $d^k \cdot n^{\mathcal{O}(1)}$. The interest in $\mathcal{F}$-COMPLETION problems started to increase with the advance of kernelization. It appeared that from the perspective of kernelization, even for the case of finite families $\mathcal{F}$, the problem is far from trivial. Guo [13] initiated the study of kernelization algorithms for $\mathcal{F}$-COMPLETION in the case when the forbidden set $\mathcal{F}$ contains the graph C_4, see Figure 1. (In fact, Guo considered edge deletion problems, but they are polynomial time equivalent to completion problems to the complements of the forbidden induced subgraphs.) In the literature, the most studied graph classes containing no induced C_4 are the *split* graphs, i.e., graphs containing no induced subgraph from $\{2K_2, C_4, C_5\}$, *threshold* graphs, i.e., $\{2K_2, P_4, C_4\}$-free graphs, and $\{C_4, P_4\}$-free, that is, *trivially perfect* graphs [3, 20]. Guo obtained polynomial kernels for the completion problems for chain graphs, split graphs, threshold graphs and trivially perfect graphs and concluded that, as a consequence of his polynomial kernelization, the corresponding $\mathcal{F}$-COMPLETION problems: CHAIN COMPLETION, SPLIT COMPLETION, THRESHOLD COMPLETION and TRIVIALLY PERFECT COMPLETION are solvable in times $\mathcal{O}(2^k + mnk)$, $\mathcal{O}(5^k + m^4n)$, $\mathcal{O}(4^k + kn^4)$, and $\mathcal{O}(4^k + kn^4)$, respectively.

The work on kernelization of $\mathcal{F}$-COMPLETION was continued by Kratsch and Wahlström [18] who showed that there exists a set $\mathcal{F}$ consisting of one graph on seven vertices for which $\mathcal{F}$-COMPLETION does not admit a polynomial kernel. Guillemot et al. [12] showed that COGRAPH

Completion, i.e., the case $\mathcal{F} = \{P_4\}$, admits a polynomial kernel, while for $\mathcal{F} = \{\overline{P_{13}}\}$, the complement of a path on 13 vertices, $\mathcal{F}$-Completion has no polynomial kernel. These results were significantly improved by Cai [5]: For $\mathcal{F} = \{P_\ell\}$ or $\mathcal{F} = \{C_\ell\}$, the problems $\mathcal{F}$-Completion and $\mathcal{F}$-Edge Deletion admit a polynomial kernel if and only if the forbidden graph has at most three edges.

Obstruction set $\mathcal{F}$	Graph class name	Complexity
$C_4, C_5, C_6, \ldots$	Chordal	SUBEPT [9]
C_4, P_4	Trivially Perfect	SUBEPT (Theorem 1)
$2K_2, C_4, C_5$	Split	SUBEPT [11]
$2K_2, C_4, P_4$	Threshold	SUBEPT (Theorem 2)
$2K_2, C_4$	Pseudosplit	SUBEPT (Theorem 3)
$\overline{P_3}, K_t,\, t = o(k)$	Co-t-cluster	SUBEPT [8]
$\overline{P_3}$	Co-cluster	E [17]
$2K_2$	$2K_2$-free	E (Theorem 4)
C_4	C_4-free	E (Theorem 5)
P_4	Cograph	E (Theorem 6)
$2K_2, P_4$	Co-Trivially Perfect	E (Theorem 7)

Figure 2: Known subexponential complexity of $\mathcal{F}$-Completion for different sets $\mathcal{F}$. SUBEPT means the problem is solvable in subexponential time $2^{o(k)}n^{\mathcal{O}(1)}$ and E means that the problem is not solvable in subexponential time unless ETH fails.

It appeared recently that for some choices of $\mathcal{F}$, $\mathcal{F}$-Completion is solvable in *subexponential* time. The exploration of this phenomenon is the main motivation for our research on this problem. The last chapter of Flum and Grohe's seminal book on parameterized complexity theory [7, Chapter 16] concerns subexponential fixed parameter tractability, the complexity class SUBEPT, which, loosely speaking—we skip here some technical conditions—is the class of problems solvable in time $2^{o(k)}n^{\mathcal{O}(1)}$, where n is the input length and k is the parameter. Until recently, the only notable exceptions of problems in SUBEPT were problems on planar graphs, and more generally, on graphs excluding some fixed graph as a minor [6]. In 2009, Alon et al. [1] used a novel application of color coding, dubbed *chromatic coding*, to show that parameterized Feedback Arc Set in Tournaments is in SUBEPT. As Flum and Grohe [7] observed, for most of the natural parameterized problems, already the classical NP-hardness reductions can be used to refute the existence of subexponential parameterized algorithms, unless the following well-known complexity hypothesis formulated by Impagliazzo, Paturi, and Zane [14] fails.

Exponential Time Hypothesis (ETH). There exists a positive real number s such that 3Sat with n variables and m clauses cannot be solved in time $2^{sn}(n+m)^{\mathcal{O}(1)}$.

Thus, it is most likely that the majority of parameterized problems are not solvable in subexponential parameterized time and until very recently no parameterized problem solvable in subexponential parameterized time on general graphs was known. A subset of the authors recently showed that Minimum Fill-in, also known as Chordal Completion, which is equivalent to $\mathcal{F}$-Completion with $\mathcal{F}$ consisting of cycles of length at least four, is in SUBEPT [9], simultaneously establishing that Chain Completion is solvable in subexponential time. Ghosh et al. [11] showed that Split Completion is solvable in subexponential time. On the other hand, Komusiewicz and Uhlmann [17], showed that an edge modification problem known as Cluster Deletion, does not belong to SUBEPT unless ETH fails. Let us note that Cluster Deletion is a special case of $\mathcal{F}$-Completion, when $\mathcal{F} = \{\overline{P_3}\}$, the complement of the path P_3. On the other hand, it is interesting to note that by the result of Fomin et al. [8], Cluster Deletion into t Clusters, i.e., the complement problem for $\mathcal{F}$-Completion for $\mathcal{F} = \{\overline{P_3}, K_t\}$, is in SUBEPT for $t = o(k)$.

Our results. In this work we extend the class of $\mathcal{F}$-Completion problems admitting subexponential time algorithms, see Figure 2. Our main algorithmic result is the following:

> The $\mathcal{F}$-Completion problem for $\mathcal{F} = \{C_4, P_4\}$ is solvable in time $2^{\mathcal{O}(\sqrt{k}\log k)}n^{\mathcal{O}(1)}$ and is thus in SUBEPT. This problem is known as Trivially Perfect Completion.

On a very high level, our algorithm is based on the same strategy as the algorithm for completion into chordal graphs [9]. As in that algorithm, we enumerate in parameterized subexponential time special structures called *potential maximal cliques* which are the maximal cliques in some minimal completion into a trivially perfect graph that uses at most k edges. As far as we succeed in enumerating these objects, we do dynamic programming to find an optimal completion. But here the similarities end. To enumerate potential maximal cliques for trivially perfect graphs, we have to use completely different structural properties from what are used for the case of chordal graphs.

We also show that within the same running time the $\mathcal{F}$-Completion problem is solvable for $\mathcal{F} = \{2K_2, C_4\}$, and $\mathcal{F} = \{2K_2, P_4, C_4\}$. This corresponds to completion into threshold and pseudosplit graphs, respectively. Let us note that combined with the results of Fomin and Villanger [9] and Ghosh et al. [11], this implies that all four problems considered by Guo in [13] are in SUBEPT, in addition to admitting a polynomial kernel. We finally complement our algorithmic findings by showing the following:

> For $\mathcal{F} = \{2K_2\}$, $\mathcal{F} = \{C_4\}$, $\mathcal{F} = \{P_4\}$ and $\mathcal{F} = \{2K_2, P_4\}$, the $\mathcal{F}$-Completion problem cannot be solved in time $2^{o(k)}n^{\mathcal{O}(1)}$ unless ETH fails.

Organization of the paper. In Section 2, we give some preliminaries, and in Section 3 we obtain structural results about trivially perfect graphs which will be used in our algorithm. This section also contains the algorithm solving Trivially Perfect Completion in subexponential time. We also prove that two other related problems are solvable in subexponential time, Threshold Completion (in Section 4) and Pseudosplit Completion (in Section 5).

The rest of the paper consists of Section 6, which gives lower bounds on $\mathcal{F}$-Completion when $\mathcal{F}$ is $\{2K_2\}$, $\{C_4\}$, $\{P_4\}$, or $\{2K_2, P_4\}$, which are completion problems to graph classes related to the trivially perfect graphs, and Section 7, where we gather some concluding remarks and state some interesting remaining questions.

2 Preliminaries

Graphs and graph classes. We consider only finite simple undirected loopless graphs. We use n to denote the number of vertices and m the number of edges in a graph G. If $G = (V, E)$ is a graph, and $A, B \subseteq V$, we write $E(A, B)$ for the edges with one endpoint in A and the other in B, and we write $E(A) = E(A, A)$ for the edges inside A. We will also write m_A to mean the number of edges inside A, i.e., $m_A = |E(A)|$.

We write $N(U)$ for $U \subseteq V(G)$ to denote the open neighborhood $\bigcup_{v \in U}(N(v)) \setminus U$, and $N[U] = N(U) \cup U$. For a graph G and a set of edges S, we write $G + S = (V, E \cup S)$ and if $U \subseteq V$ is a set of vertices, $G - U = G[V \setminus U]$. A *universal vertex* in a graph is a vertex v such that $N[v] = V(G)$. Let $\mathrm{uni}(G)$ denote the set of universal vertices of G. Observe that $\mathrm{uni}(G)$ is always a clique, and we will refer to it as the *(maximal) universal clique*.

If $\mathcal{F}$ is a set of graphs, we denote by $\mathcal{G}_\mathcal{F}$ the graph class characterized by $\mathcal{F}$ as forbidden induced subgraphs, i.e.,

$$\mathcal{G}_\mathcal{F} = \{G \mid \text{for all } F \in \mathcal{F},\ G \text{ does not have } F \text{ as an induced subgraph}\}.$$

Observe that for any $\mathcal{F}$, the class $\mathcal{G}_\mathcal{F}$ is hereditary.

The complexity classes FPT and SUBEPT, and polynomial kernels. A (parameterized) language $\mathcal{L} \subseteq \Sigma^* \times \mathbb{N}$ for a finite alphabet Σ is contained in FPT if there is a computable function $f\colon \mathbb{N} \to \mathbb{N}$ such that any instance (Q, k) can be decided with respect to $\mathcal{L}$ in time $f(k) \cdot p(|Q|)$ for some polynomial function $p\colon \mathbb{N} \to \mathbb{N}$. We call such a problem *fixed-parameter tractable*.

If a parameterized language $\mathcal{L}$ can be decided in time $2^{o(k)} \cdot p(|Q|)$ for some polynomial function p we say that $\mathcal{L}$ is in SUBEPT. Clearly SUBEPT is contained in FPT.

A *kernelization algorithm* for a parameterized language $\mathcal{L} \subseteq \Sigma^* \times \mathbb{N}$ is an algorithm that on input $(Q, k) \in \Sigma^* \times \mathbb{N}$ in time polynomial in $|Q| + k$, outputs a pair $(Q', k') \in \Sigma^* \times \mathbb{N}$, called a *kernel*, such that $(Q, k) \in \mathcal{L}$ if and only if $(Q', k') \in \mathcal{L}$, $|Q'| \leq g(k)$ for some computable function $g\colon \mathbb{N} \to \mathbb{N}$, and $k' \leq k$. If g is a polynomial, we say that $\mathcal{L}$ admits a *polynomial kernel*.

3 Completion to trivially perfect graphs

In this section we study the Trivially Perfect Completion problem which is the special case of $\mathcal{F}$-Completion for $\mathcal{F} = \{C_4, P_4\}$. The decision version of the problem was shown to be NP-complete by Yannakakis [24]. Since trivially perfect graphs are characterized by a finite set of forbidden induced subgraphs, it follows from Cai [4] that the problem also belongs to the class FPT.

The main result of this section is the following theorem.

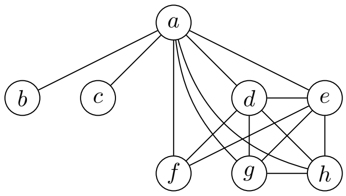
Theorem 1. *For an input* (G, k)*,* Trivially Perfect Completion *is solvable in time* $2^{\mathcal{O}(\sqrt{k}\log k)} + \mathcal{O}(kn^4)$*.*

Throughout this section, an edge set S is called a *completion* for G if $G + S$ is trivially perfect. Furthermore, a set S is called a *minimal completion* for G if no proper subset of S is a completion for G. The main outline of the algorithm is as follows:

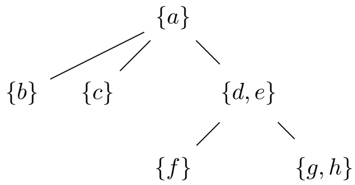
Step A: On input (G, k), we first apply the algorithm by Guo [13] to obtain a kernel of size $\mathcal{O}(k^3)$. The running time of this algorithm is $\mathcal{O}(kn^4)$.

Step B: Assuming our input instance is of size $\mathcal{O}(k^3)$, we show how to generate all special vertex subsets of the kernel which we call *vital potential maximal cliques* in time $2^{\mathcal{O}(\sqrt{k}\log k)}$. A vital potential maximal clique $\Omega \subseteq V(G)$ is a vertex subset which is a maximal clique in some minimal completion and the subgraph induced by Ω can be turned into a complete graph by adding at most k edges.

Step C: Using dynamic programming, we show how to compute an optimal solution or to conclude that (G, k) is a ***no*** instance, in time polynomial in the number of vital potential maximal cliques.



(a) A trivially perfect graph.



(b) Universal clique decomposition with bags as labels.

Label (bag)	Block	Tail
$\{a\}$	$(\{a\}, V)$	$\{a\}$
$\{b\}$	$(\{b\}, \{b\})$	$\{a, b\}$
$\{c\}$	$(\{c\}, \{c\})$	$\{a, c\}$
$\{d, e\}$	$(\{d, e\}, \{d, e, f, g, h\})$	$\{a, d, e\}$
$\{f\}$	$(\{f\}, \{f\})$	$\{a, d, e, f\}$
$\{g, h\}$	$(\{g, h\}, \{g, h\})$	$\{a, d, e, g, h\}$

(c) Table of bags with corresponding blocks and tails.

Figure 3: In the first figure, we have a trivially perfect graph, and in the second, a *universal clique decomposition* of the graph with the bags as labels. Finally we have a table of the bags and the corresponding blocks and tails. Notice that for a block (B, D) and tail Q, $B \subseteq D$ and $B \subseteq Q$. Furthermore, in any leaf block it holds that $B = D$, and in the root block it holds that $D = V$.

3.1 Structure of trivially perfect graphs

Apart from the aforementioned characterization by forbidden induced subgraphs, several other equivalent definitions of trivially perfect graphs are known. These definitions reveal more structural properties of this graph class which will be essential in our algorithm. Therefore, before proceeding with the proof of Theorem 1, we establish a number of results on the structure of trivially perfect graphs and minimal completions which will be useful in the proof of Theorem 1.

The following proposition is often used as an alternative definition of trivially perfect graphs.

Proposition 3.1 ([15])**.** *The class of trivially perfect graphs can be defined recursively as follows:*

- K_1 *is a trivially perfect graph.*
- *Adding a universal vertex to a trivially perfect graph results in a trivially perfect graph.*
- *The disjoint union of two trivially perfect graphs results in a trivially perfect graph.*

The trivially perfect graphs have a decomposition tree which we call a *universal clique decomposition*, in which each node in the tree corresponds to a maximal set of vertices that all are universal for the graph induced by the vertices in the subtree.

Let T be a rooted tree and t be a node of T. We denote by T_t the maximal subtree of T rooted in t. We can now use the universal clique uni(G) of a trivially perfect graph $G = (V, E)$ to make a decomposition structure.

Definition 3.2 (Universal clique decomposition)**.** A *universal clique decomposition* of a connected trivially perfect graph $G = (V, E)$ is a pair $(T = (V_T, E_T), \mathcal{B} = \{B_t\}_{t \in V_T})$, where T is a rooted tree and $\mathcal{B}$ is a partition of the vertex set V into disjoint nonempty subsets, such that

- if $vw \in E(T)$ and $v \in B_t, w \in B_s$, then either $t = s$, t is an ancestor of s in T, or s is an ancestor of t in T, and
- for every node $t \in V_T$, the set of vertices B_t is the maximal universal clique in the subgraph induced by the vertices of G in the bags corresponding to T_t.

We call the vertices of T *nodes* and the sets in $\mathcal{B}$ *bags* of the universal clique decomposition $(T, \mathcal{B})$. By slightly abusing the notations, we often do not distinguish between nodes and bags. Note that by the definition, in a universal clique decomposition every non-leaf node has at least two children.

Lemma 3.3. *A connected graph G admits a universal clique decomposition if and only if it is trivially perfect. Moreover, such a decomposition is unique.*

Proof. From right to left, we proceed by induction on the number of vertices using Proposition 3.1. The base case is when we have one vertex, K_1 which is a trivially perfect graph and also admits a unique universal clique decomposition. The induction step is when we add a vertex v, and by the definition of trivially perfect graphs, v is a universal vertex. Either we add a universal vertex to a connected trivially perfect graph, in which case we simply add the vertex to the root bag, or we add a universal vertex to two or more trivially perfect graphs. In this case, we create a new tree, with r_v being the root, and each of the trees for the graphs subtrees below r_v. Since v is the only universal vertex in the graph, the constructed structure is a universal clique decomposition. Observe that the constructed decompositions are unique (up to isomorphisms).

From left to right, we proceed by induction on the height of the universal clique decomposition. Suppose $(T, \mathcal{B})$ is a universal clique decomposition of a graph G. Consider the case when T has height 1, i.e., we have only one single tree node (and one bag). Then this bag, by Proposition 3.1, is a clique (every vertex in the bag is universal), and since a complete graph is trivially perfect, the base case holds. Consider now the case when T has height at least 2. Let r be the root of T, and let $x_1, x_2, \ldots, x_p$ be children of r in T. Observe that tree T_{x_i} is a universal clique decomposition for the graph $G[\bigcup_{t \in V(T_{x_i})} B_t]$ for each $i = 1, 2, \ldots, p$. Hence, by the induction hypothesis we have that $G[\bigcup_{t \in V(T_{x_i})} B_t]$ is trivially perfect. To see that G is trivially perfect as well, observe that G can be obtained by taking the disjoint union of graphs $G[\bigcup_{t \in V(T_{x_i})} B_t]$ for $i = 1, 2, \ldots, p$, and adding $|B_r|$ universal vertices.

□

For the purposes of the dynamic programming used in the algorithm, we define the following notion.

Definition 3.4 (Block)**.** Let $(T = (V_T, E_T), \mathcal{B} = \{B_t\}_{t \in V_T})$ be the universal clique decomposition of a connected trivially perfect graph G. For each node $t \in V_t$, we associate a *block* $L_t = (B_t, D_t)$, where

- B_t is the subset of V contained in the bag corresponding to t, and
- D_t is the set of vertices of V contained in the bags corresponding to the nodes of the subtree T_t.

The *tail* of a block L_t is the set of vertices Q_t contained in the bags corresponding to the nodes of the path from t to r in T, where r is the root of T.

When t is a leaf of T, we call the block $L_t = (B_t, D_t)$ a *leaf block*, and if t is the root, we call L_t the *root block*.

Observe that for every block $L_t = (B_t, D_t)$ with tail Q_t we have that $B_t \subseteq Q_t$, $B_t \subseteq D_t$, and $D_t \cap Q_t = B_t$, see Figure 3. Note also that Q_t is a clique and the vertices of Q_t are universal to $D_t \setminus B_t$.

The following lemma summarizes the properties of universal clique decompositions, maximal cliques, and blocks used in our proof.

Lemma 3.5. *Let $(T, \mathcal{B})$ be the universal clique decomposition of a connected trivially perfect graph G and let $L_t = (B_t, D_t)$ be a block with Q_t its tail.*

(i) If L_t is a leaf block and Q_t its tail, then $Q_t = N_G[v]$ for every $v \in B_t$.

(ii) The following are equivalent:

(1) L_t is a leaf block,

(2) $D_t = B_t$, and

(3) Q_t is a maximal clique of G.

(iii) If L_t is a non-leaf block, then for every two vertices u, v from different connected components of $G[D_t \setminus B_t]$, we have that $Q_t = N_G(u) \cap N_G(v)$.

Proof. *(i)* Since Q_t is a clique, we have that $Q_t \subseteq N_G[v]$. On the other hand, since $v \in B_t$ and t is a leaf block, we have that $Q_t \supseteq N_G[v]$ by the definition of universal clique decomposition.

(ii) We prove the chain (1) $\to$ (2) $\to$ (3) $\to$ (1). Since L_t is a leaf block, and D_t is the set of vertices in the bags in the subtree rooted at L_t, $B_t = D_t$. Then by (i) we have that $N_G[v] = Q_t$ for any $v \in B_t$; hence Q_t is maximal. Finally, if Q_t is a maximal clique in the graph, i.e., it cannot be extended, by definition t cannot have any children so t must be a leaf block.

(iii) Suppose $L_t = (B_t, D_t)$ is a non-leaf block and D_1 and D_2 are two connected components of $G_t = G[D_t \setminus B_t]$. Let $v \in D_1$ and $u \in D_2$ and observe that since they are in different connected components of $G[D_t \setminus B_t]$, $N_{G_t}[v] \cap N_{G_t}[u] = \emptyset$. By the universality of Q_t, the result follows; $Q_t = N[v] \cap N[u]$. ☐

3.2 Structure of minimal completions

Before we proceed with the algorithm, we provide some properties of minimal completions. The following lemma gives insight to the structure of a ***yes*** instance.

Lemma 3.6. *Let $G = (V, E)$ be a connected graph, S a minimal completion and $H = G + S$. Suppose $L = (B, D)$ is a block in some universal clique decomposition of H and denote by $D_1, D_2, \ldots, D_\ell$ the connected components of $H[D] - B$.*

(i) If L is not a leaf block, then $\ell > 1$,

(ii) If $\ell > 1$, then in G every vertex $v \in B$ has at least one neighbor in each set $D_1, D_2, \ldots, D_\ell$,

(iii) the graph $G[D_i]$ is connected for every $i \in \{1, \ldots, \ell\}$, and

(iv) for every $i \in \{1, \ldots, \ell\}$, $B \subseteq N_G(D \setminus (B \cup D_i))$.

Proof. We prove this case by case. *(i)* Let (B, D) be a non-leaf block. Since B is maximal, D is not a clique, so by the recursive definition of trivially perfect graphs, $H[D] - B$ is the disjoint union of two or more trivially perfect graphs, hence $\ell > 1$.

(ii) We distinguish between two cases, when $\ell = 2$ and when $\ell > 2$. Suppose $\ell = 2$ and furthermore assume towards a contradiction (without loss of generality) that $v \in B$ has no

neighbor in D_2 in G. Consider the universal clique decomposition where v is in D_1 instead of in B. Since D_2 is non-empty, S contains edges of the type vu with $u \in D_2$. Let $S' = S \setminus E_H(\{v\}, D_2)$. We can observe that $G + S'$ is trivially perfect with the same decomposition except that v is in D_1 and not in B. This contradicts the assumption that S was a minimal completion.

Suppose now that $\ell > 2$ and that there are sets (again without loss of generality) $D_1, \ldots, D_p$ for which $v \in B$ does not have neighbors in G. Consider the universal clique decomposition where v is a singleton bag B' below B, and $D_{p+1}, D_{p+2}, \ldots, D_\ell$ are all connected to B' instead of B. This is indeed a universal clique decomposition of the graph $G + S'$ where, as in the previous case, $S' = S \setminus E_H(\{v\}, D_1 \cup D_2 \cup \cdots \cup D_p)$. We have again reached a contradiction to the minimality of S.

(iii) For the sake of a contradiction, suppose $G[D_a]$ was disconnected. Let (D_{a_1}, D_{a_2}) be a partition of D_a such that there is no edge between D_{a_1} and D_{a_2} in G. Clearly, $H[D_{a_1}]$ and $H[D_{a_2}]$ are trivially perfect graphs as induced subgraphs of H, hence they admit some universal clique decompositions. Since $H[D_a]$ is connected, we infer that S contains some edges between D_{a_1} and D_{a_2}. Let now $S' = S \setminus \{uv \mid u \in D_{a_1}, v \in D_{a_2}, uv \in S\}$; by the previous argument we have that $S' \subsetneq S$. Modify now the given universal clique decomposition of H by removing the subtree below B that corresponds to D_a, and attaching instead two subtrees below B that are universal clique decompositions of $H[D_{a_1}]$ and $H[D_{a_2}]$. Observe that thus we obtain a universal clique decomposition of $G + S'$, which shows that $G + S'$ is trivially perfect. This is a contradiction with the minimality of S.

(iv) Follows directly from *(ii)* as every vertex of B has edges in G to two different connected components of $D \setminus B$. □

3.3 Algorithm

As has been mentioned several times, the following concept is crucial for our algorithm. Recall from Section 2 that when Ω is a set of vertices in a graph G, by m_Ω we mean the number of edges in $G[\Omega]$.

Definition 3.7 (Vital potential maximal clique)**.** Let (G, k) be an instance of Trivially Perfect Completion. A vertex set $\Omega \subseteq V(G)$ is a *potential maximal clique* if Ω is a maximal clique in some minimal trivially perfect completion of G. A potential maximal clique Ω is referred to as *vital* if $\binom{|\Omega|}{2} - m_\Omega \leq k$, i.e., if $G[\Omega]$ has at most k non-edges.

Observe that given a ***yes*** instance (G, k) and a minimal completion S of size at most k, every maximal clique in $G + S$ is a vital potential maximal clique in G.

The following definition will be useful:

Definition 3.8 (Fill number)**.** Let $G = (V, E)$ be a graph, S a completion and $H = G + S$. We define the *fill* of a vertex v, denoted by $\mathrm{fn}_H^G(v)$ as the number of edges incident to v in S.

Let us observe that there are at most $2\sqrt{k}$ vertices v such that $\mathrm{fn}_H^G(v) > \sqrt{k}$. It follows that for every set $U \subseteq V$ such that $|U| > 2\sqrt{k}$, there is a vertex $u \in U$ with $\mathrm{fn}_H^G(u) \leq \sqrt{k}$. Any vertex u such that $\mathrm{fn}_H^G(u) \leq \sqrt{k}$ will be referred to as a *cheap* vertex.

Everything is settled to start the proof of Theorem 1. The proof is constructive, and we give an algorithm performing in three steps. We compress the instance to an instance of size $\mathcal{O}(k^3)$, we then enumerate all (subexponentially many) vital potential cliques in this new instance, and finally we do a dynamic programming procedure on these objects.

Step A. Kernelization. For a given input (G, k), we start by applying the kernelization algorithm by Guo [13] to construct in time $\mathcal{O}(kn^4)$ an equivalent instance (G', k'), where G' has

$\mathcal{O}(k^3)$ vertices and $k' \leq k$. From now on we will assume that the graph G has $\mathcal{O}(k^3)$ vertices. Without loss of generality, from now on we also assume that G is connected, since we may treat each connected component of G separately.

Step B. Enumeration. In this step, we give an algorithm that in time $2^{\mathcal{O}(\sqrt{k}\log k)}$ outputs a family $\mathcal{C}$ of vertex subsets of G such that

- the size of $\mathcal{C}$ is $2^{\mathcal{O}(\sqrt{k}\log k)}$, and
- every vital potential maximal clique belongs to $\mathcal{C}$.

We identify five different types of vital potential maximal cliques. For each type i, $1 \leq i \leq 5$, we list a family $\mathcal{C}_i$ of $2^{\mathcal{O}(\sqrt{k}\log k)}$ subsets containing all vital potential maximal cliques of this type. Finally, $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_5$. We show that every vital potential maximal clique of (G, k) is of at least one type and that all objects of each type can be enumerated in $2^{\mathcal{O}(\sqrt{k}\log k)}$ time.

Let Ω be a vital potential maximal clique. By the definition of Ω, if (G, k) is a ***yes*** instance, then there is a minimal completion with at most k edges into a trivially perfect graph H such that Ω is a maximal clique in H. Let $(T = (V_T, E_T), \mathcal{B} = \{B_t\}_{t\in V_T})$ be the universal clique decomposition of a trivially perfect graph. Recall that by Lemma 3.5, Ω corresponds to a path $P_{rt} = B_{t_0}B_{t_1}\cdots B_{t_q}$ in T from the root $r = t_0$ to a leaf $t = t_q$. Then for the corresponding leaf block (B_t, D_t) with tail Q_t, we have that $\Omega = Q_t$. To simplify the notation, we use B_i for B_{t_i}.

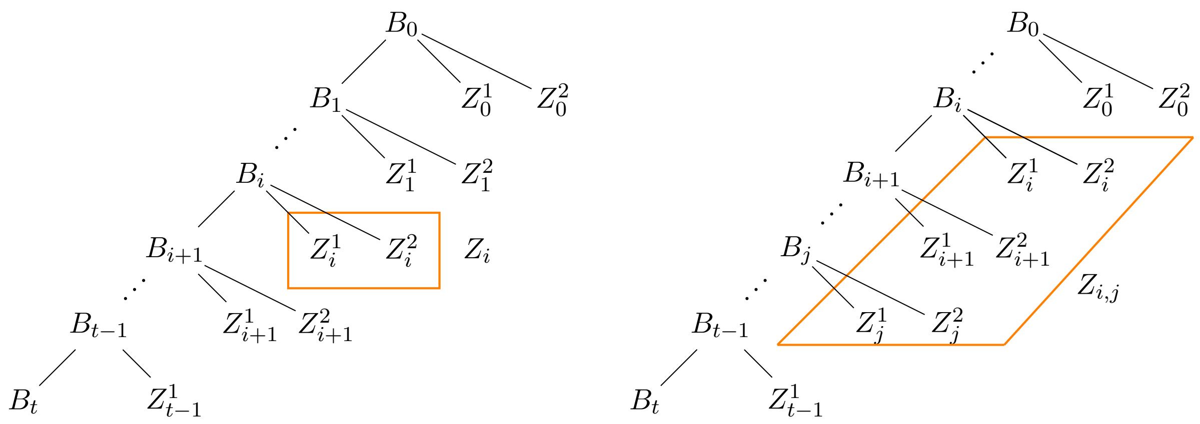
Type 1. Potential maximal cliques of the first type are such that $|V \setminus \Omega| \leq 2\sqrt{k} + 2$. The family $\mathcal{C}_1$ consists of all sets $W \subseteq V$ such that $|V \setminus W| \leq 2\sqrt{k} + 2$. There are $\binom{\mathcal{O}(k^3)}{2\sqrt{k}+2}$ such sets and we can find all of them in time $2^{\mathcal{O}(\sqrt{k}\log k)}$ by the brute-force algorithm trying all vertex subsets of size at least $|V| - 2\sqrt{k} + 2$. Thus every Type 1 vital potential maximal clique is in $\mathcal{C}_1$.

Type 2. By Lemma 3.5 *(i)*, there is a leaf block (B_t, D_t) with tail Q_t such that $\Omega = Q_t = N_H[v]$ for each vertex $v \in D_t = B_t$. Vital potential cliques of the second type are such that $|B_t| > 2\sqrt{k}$. If (G, k) is a ***yes*** instance, then at least one vertex $v \in B_t$ should be *cheap*, i.e., $\text{fn}_H^G(v) \leq \sqrt{k}$. We generate the family $\mathcal{C}_2$ as follows. Every set in $\mathcal{C}_2$ is of the form $W_1 \cup W_2$, where $W_1 = N_G[v]$ for some $v \in V$, and $|W_2| \leq \sqrt{k}$. There are at most $\mathcal{O}(\binom{\mathcal{O}(k^3)}{\sqrt{k}}k^3)$ such sets and they can be enumerated by computing for every vertex v the set $W_1 = N_G[v]$ and adding to each such set all possible subsets of size at most $\sqrt{k}$. Hence every Type 2 vital potential maximal clique is in $\mathcal{C}_2$.

Thus if Ω is not of Types 1 or 2, then $|V \setminus \Omega| > 2\sqrt{k} + 2$ and for the corresponding leaf block, we have $|B_t| \leq 2\sqrt{k}$. Since $|V \setminus \Omega| > 2\sqrt{k} + 2$ it follows that if (G, k) is a ***yes*** instance, then $V \setminus \Omega$ contains at least two cheap vertices, i.e., vertices with fill number at most $\sqrt{k}$.

We partition the nodes of T that are not on the path $B_0, B_1, \ldots, B_t$ into t sets $Z_0, Z_1, \ldots, Z_{t-1}$ according to the nodes of the path P_{rt}. Node $x \notin V(P_{rt})$ belongs to Z_i, $i \in \{0, \ldots, t-1\}$, if i is the largest integer such that B_i is an ancestor of x in T. In other words, Z_i consists of bags of subtrees outside P_{rt} growing from B_i. See Figure 4a.

Let j be the maximum index such that a bag from Z_j contains a cheap vertex. We define the set of vertices $Z_{>j} = \bigcup_{i=j+1}^{t-1} Z_i$. Observe that since $Z_{>j}$ does not contain cheap vertices, $|Z_{>j}| \leq 2\sqrt{k}$. We also define $V_{0,j}$ as the set of vertices contained in the bags corresponding to nodes $B_0, B_1, \ldots, B_j$ of P_{rt} and set $V_{j+1,t}$ as the set of vertices contained in bags $B_{j+1}, \ldots, B_t$. Observe also that $\Omega = V_{0,j} \cup V_{j+1,t}$ and by the definition of a block, $V_{0,j}$ is exactly the tail Q_j of the block (B_j, D_j). From Lemma 3.6 *(i, iv)* we have that $V_{j+1,t} \subseteq B_t \cup N_G(Z_{>j}) \subseteq \Omega$. This follows from the fact that every vertex in $\Omega \setminus B_t$ has at least one neighbor in G in Z_ℓ for some ℓ.



(a) Z_i for vital potential maximal clique $\Omega = B_1 \cup B_2 \cup \cdots \cup B_i \cup \cdots \cup B_t$. Z_i is the collection of vertices corresponding to bags below B_i but not below B_{i+1}.

(b) The set $Z_{i,j}$ for $i \leq j$ are the vertices belonging to bags in the subtree T_i except Ω, the maximal clique, and vertices in bags of T_{j+1}.

Figure 4: Illustration of sets outside the maximal clique we use to find the maximal clique. The figure shows a universal clique decomposition of a completion where $\Omega = B_0 \cup \ldots \cup B_t$ is a maximal clique. Observe that the leaf block is $L_t = (B_t, B_t)$ and that its tail is $Q_t = \Omega$.

Let v be a cheap vertex belonging to Z_j. The remaining types of vital potential maximal cliques are defined according to the existence and locations of a few other cheap vertices in T. We use C^v to denote the connected component of $G[D_j] - B_j$ containing v.

Type 3. For vital potential maximal cliques of this type there is a cheap vertex $u \neq v$ belonging to Z_j but not belonging to C^v. Since $V_{0,j} = Q_j$, by Lemma 3.5 *(iii)*, we have that $V_{0,j} = N_H(u) \cap N_H(v)$ and $V_{j+1,t} \subseteq B_t \cup N_G(Z_{>j}) \subseteq \Omega$. Hence we arrive at

$$\Omega = V_{0,j} \cup V_{j+1,t} = \big(N_H(u) \cap N_H(v)\big) \cup B_t \cup N_G(Z_{>j}).$$

Family $\mathcal{C}_3$ consists of all sets of type $W_1 \cup W_2 \cup W_3$, where:

- $|W_1| \leq 2\sqrt{k}$. Enumerating sets W_1 corresponds to guessing B_t.
- W_2 is the open neighborhood in G of a set of size at most $2\sqrt{k}$. The set W_2 corresponds to $N_G(Z_{>j})$.
- W_3 is the intersection of the sets $N_G(x) \cup A$ and $N_G(y) \cup B$, where $x, y \in V$, and A, B are sets of size at most $\sqrt{k}$. The set W_3 corresponds to intersection of two neighborhoods in H of two cheap vertices.

It is clear that the size of the family $\mathcal{C}_3$ is $2^{\mathcal{O}(\sqrt{k}\log k)}$ and that all sets from $\mathcal{C}_3$ can be listed using $2^{\mathcal{O}(\sqrt{k}\log k)}$ time. It follows from the construction that every Type 3 vital potential maximal clique is in $\mathcal{C}_3$.

Type 4. Let Z be the set of vertices of $V \setminus \Omega$ which do not belong to C^v. In other words, $Z = (V \setminus \Omega) \setminus V(C^v)$. Vital potential maximal cliques of Type 4 are such that Z contains no cheap vertices. Thus the only cheap vertices among vertices of $V \setminus \Omega$ belong to C^v. In this case, we have that $|Z| \leq 2\sqrt{k}$.

Recall that $\Omega = V_{0,j} \cup V_{j+1,t}$, where $V_{0,j}$ and $V_{j+1,t}$ are the vertices contained in bags of paths from r to B_j, and correspondingly, from B_{j+1} to t in T. By Lemma 3.6, we have that

$V_{j+1,t} = (B_t \cup N_G(Z_{>j})) \setminus N_H(v)$. Furthermore, $V_{0,j} = N_G(Z \cup V_{j+1,t})$, so it follows that

$$\begin{aligned}\Omega &= V_{0,j} \cup V_{j+1,t} \\ &= \big(N_G(Z \cup ((B_t \cup N_G(Z_{>j})) \setminus N_H(v)))\big) \cup \big((B_t \cup N_G(Z_{>j})) \setminus N_H(v)\big).\end{aligned}$$

We therefore let the family $\mathcal{C}_4$ consist of all sets of $W_1 \cup W_2$, where

- $W_1 = (X_1 \cup N_G(X_2)) \setminus (N_G(v) \cup X_3)$ and the sets X_1, X_2, and X_3 are sets of size at most $2\sqrt{k}$ and $v \in V$. The set W_1 corresponds to guessing $V_{j+1,t}$, X_1 to B_t, X_2 to $Z_{>j}$, and $N_G(v) \cup X_3$ to $N_H(v)$, and
- $W_2 = N_G(X_4 \cup W_1)$, where X_4 is of size at most $2\sqrt{k}$ and corresponds to guessing Z.

By the construction, the size of $\mathcal{C}_4$ is $2^{\mathcal{O}(\sqrt{k}\log k)}$ and all sets from $\mathcal{C}_4$ can be listed in time $2^{\mathcal{O}(\sqrt{k}\log k)}$. It also follows from the construction that every Type 4 vital potential maximal clique is in $\mathcal{C}_4$.

Type 5. The only remaining type of vital potential maximal cliques are such that a cheap vertex $u \neq v$ is in Z. If Ω is not of Type 3, then we know that at least one cheap vertex is in some bag of Z_i, $i < j$. Let $j' < j$ be the largest index smaller than j such that $Z_{j'}$ contains a cheap vertex. Let u be such a vertex.

Let $V_{0,j'}$ be the set of vertices contained in the $B_0, B_1, \ldots, B_{j'}$. Then $V_{0,j'} = Q_{j'}$ and by Item *(iii)* of Lemma 3.5, $V_{0,j'} = N_H(u) \cap N_H(v)$. Let

$$Z' = \bigcup_{i=j'+1}^{j} Z_i \setminus C^v.$$

There is no cheap vertex in Z', hence $|Z'| \leq 2\sqrt{k}$. On the other hand, by Item *(iv)* of Lemma 3.6, $V_{j'+1,j}$, that is, vertices contained in the bags $B_{j'+1}, \ldots, B_j$, is contained in $N_G(V_{j+1,t} \cup Z_{>j}) \cup N_G(Z') \subseteq \Omega$. Thus

$$\begin{aligned}\Omega &= V_{j+1,t} \cup V_{0,j'} \cup V_{j'+1,j} \\ &= V_{j+1,t} \cup \big(N_H(u) \cap N_H(v)\big) \cup \big(N_G(V_{j+1,t} \cup Z_{>j}) \cup N_G(Z')\big).\end{aligned}$$

Finally, as in Type 4 we have that $V_{j+1,t} = (B_t \cup N_G(Z_{>j})) \setminus N_H(v)$. Therefore, we let $\mathcal{C}_5$ consist of all sets of type $W_1 \cup W_2 \cup W_3$, where

- $W_1 = (X_1 \cup N_G(X_2)) \setminus (N_G(v) \cup X_3)$ and sets X_1, X_2, and X_3 are sets of size at most $2\sqrt{k}$ and $v \in V$. As in the previous case for Type 4 vital potential maximal cliques, the set W_1 corresponds to $V_{j+1,t}$.
- $W_2 = (N_G(u) \cup X_4) \cap (N_G(v) \cup X_5)$. Here X_4, X_5, are sets of size at most $\sqrt{k}$ and $u, v \in V$. The set W_2 corresponds to $V_{0,j'}$, while $N_G(u) \cup X_4$ and $N_G(v) \cup X_5$ to $N_H(u)$ and $N_H(v)$ respectively.
- $W_3 = N_G(W_1 \cup X_2) \cup N_G(X_6)$, where X_2 and X_6 are sets of size at most $2\sqrt{k}$, and X_2 corresponds to $Z_{>j}$ and X_6 to Z'.

It is easy to check that the size of family $\mathcal{C}_5$ is $2^{\mathcal{O}(\sqrt{k}\log k)}$ and that its elements can be enumerated in the same amount of time. By the construction of $\mathcal{C}_5$, every Type 5 vital potential maximal clique is in $\mathcal{C}_5$.

To conclude, we constructed the family $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_5$ of size $2^{\mathcal{O}(\sqrt{k}\log k)}$ and every vital potential maximal clique belongs to $\mathcal{C}$. We summarize our findings with the following lemma.

Lemma 3.9 (Enumeration Lemma). *Let (G, k) be an instance of* Trivially Perfect Completion *such that $|V(G)| = \mathcal{O}(k^3)$. Then in time $2^{\mathcal{O}(\sqrt{k}\log k)}$, we can construct a family $\mathcal{C}$ consisting of $2^{\mathcal{O}(\sqrt{k}\log k)}$ subsets of $V(G)$ such that every vital potential maximal clique of (G, k) is in $\mathcal{C}$.*

Step C. Dynamic programming. At this step we assume that we have the family $\mathcal{C}$ containing all vital potential maximal cliques of (G, k). We start by generating in time $2^{\mathcal{O}(\sqrt{k}\log k)}$ a family $\mathcal{S}$ of pairs (X, Y), where $X, Y \subseteq V$, where $V = V(G)$, such that

- for every minimal completion H and the corresponding universal clique decomposition $(T, \mathcal{B})$, every block (B, D) is in $\mathcal{S}$, and
- the size of $\mathcal{S}$ is $2^{\mathcal{O}(\sqrt{k}\log k)}$.

The construction of $\mathcal{S}$ is based on the following lemmata.

Lemma 3.10. *Let G be a graph, S a minimal completion and (B, D) a non-leaf and non-root block of the universal clique decomposition of $H = G + S$, with Q being its tail. Then*

(i) Q is the intersection of two vital potential maximal cliques Ω_1 and Ω_2 of G,

(ii) $B = Q \setminus \Omega_3$ for some vital potential maximal clique Ω_3, and

(iii) D is the connected component of $G - (Q \setminus B)$ containing B.

Proof. (i) Consider two connected components D_1 and D_2 of $H[D \setminus B]$ and let Ω_1' and Ω_2' be maximal cliques in D_1 and D_2. Observe that $\Omega_1 = \Omega_1' \cup Q$ and $\Omega_2 = \Omega_2' \cup Q$ are maximal cliques in H. By definition, Ω_1 and Ω_2 are vital potential maximal cliques in G and $\Omega_1 \cap \Omega_2 = Q$.

(ii) Let $\hat{L} = (\hat{B}, \hat{D})$ be the parent block of (B, D). Since $\hat{L}$ is not a leaf-block, $\hat{L}$ has at least two children and thus there is a block (B', D') which is also a child of $\hat{L}$. By the previous point, $\hat{Q}$, the tail of $\hat{L}$ is exactly $\hat{Q} = \Omega_1 \cap \Omega_3$ for some vital potential maximal clique Ω_3. It follows that $B = Q \setminus \Omega_3$.

(iii) It follows from Lemma 3.6 that $G[D]$ is connected. Then it follows immediately that D is the unique connected component of $G - (Q \setminus B)$ containing B. ☐

Lemma 3.11. *Let G be a graph, S a minimal completion and $L = (B, D)$ a leaf block of the universal clique decomposition of $H = G + S$. If H is not a complete graph, then*

(i) $B = \Omega_1 \setminus \Omega_2$ for some vital potential maximal cliques Ω_1 and Ω_2, and

(ii) $D = B$.

Proof. (i) Let $\hat{L} = (\hat{B}, \hat{D})$ be the parent block of L, which exists since L is not the root block. Let $L' = (B', D')$ be a child of $\hat{L}$ which is not L. If $L' = (B', D')$ is a leaf, then set $L'' = L$, and if not, then let $L'' = (B'', D'')$ be a leaf having L' as an ancestor. The blocks L' and L'' exist since $\hat{L}$ is not a leaf. Furthermore, let $\hat{Q}$ be the tail of $\hat{L}$, and let $\Omega_1 = N_H[B]$ and $\Omega_2 = N_H[B'']$ be two maximal cliques in H. We know from above that $\hat{Q} = \Omega_1 \cap \Omega_2$ and hence $B = \Omega_1 \setminus \Omega_2$.

(ii) This follows immediately from Lemma 3.5. ☐

Lemma 3.12. *Let G be a connected graph, S a minimal completion and $L = (B, D)$ the root block of the universal clique decomposition of $H = G + S$. If H is not a complete graph, then*

(i) the tail of L is B,

(ii) $B = \Omega_1 \cap \Omega_2$ *for some vital potential maximal cliques* Ω_1 *and* Ω_2*, and*

(iii) $D = V$.

Proof. (i) By definition, the tail is the collection of vertices from B to the root. Since L is a root block, the tail is B itself.

(ii) This follows in the same manner as in the proof of Lemma 3.10 *(i)*, since B is the tail of block L.

(iii) From the definition of universal clique decompositions we have that D is the connected component of $H[V \setminus (Q \setminus B)]$ containing B, but $Q \setminus B = \emptyset$, hence D is the connected component of H containing B and since H is connected, the result follows. ☐

By making use of Lemmata 3.10–3.12, one can construct the required family $\mathcal{S}$ by going through all possible triples of elements of $\mathcal{C}$. The size of $\mathcal{S}$ is at most $|\mathcal{C}|^3 = 2^{\mathcal{O}(\sqrt{k}\log k)}$ and the running time of the construction of $\mathcal{S}$ is $2^{\mathcal{O}(\sqrt{k}\log k)}$. Note here that by Lemma 3.6 *(iii)* and the fact that G is connected, we may discard from $\mathcal{S}$ every pair (B, D) where $G[D]$ is not connected.

For every pair $(X, Y) \in \mathcal{S}$, with $X \subseteq Y \subseteq V$, we define $\mathrm{dp}\,(X, Y)$ to be the minimum number of edges required to add to $G[Y]$ to obtain a trivially perfect graph where X is the maximal universal clique. Thus, to compute an optimal solution, it is sufficient to go through the values $\mathrm{dp}\,(X, Y)$, where $(X, Y) \in \mathcal{S}$ with $Y = V$. In other words, to compute a minimum completion we can find

$$\min_{(X,V)\in\mathcal{S}} \mathrm{dp}\,(X, V). \tag{1}$$

We compute (1) by making use of dynamic programming over sets of $\mathcal{S}$. For every pair $(X, Y) \in \mathcal{S}$ which can be a leaf block for some completion, i.e., for all pairs with $X = Y$, we put

$$\mathrm{dp}\,(X, X) = \binom{|X|}{2} - m_X.$$

For $(X, Y) \in \mathcal{S}$ with $X \subsetneq Y$, if (X, Y) is a block of some minimal completion H, then in H, we have that X is a universal clique in $H[Y]$, every vertex of X is adjacent to all vertices of $Y \setminus X$ and the number of edges in $H[Y \setminus X]$ is the sum of edges in the connected components of $H[Y \setminus X]$. By Lemma 3.6, the vertices of every connected component Y' of $H[Y \setminus X]$ induce a connected component in $G[Y \setminus X]$. We can notice that for each connected component Y' of $H[Y \setminus X]$ the decomposition of H contains a new block (X', Y') and since $\mathcal{S}$ contains all blocks of minimal trivially perfect completions it follows that $(X', Y') \in \mathcal{S}$.

Now for $(X, Y) \in \mathcal{S}$ on increasing size of Y, we use $m_{X,Y\setminus X} = |E(X, Y \setminus X)|$ to denote the number of edges between X and $Y \setminus X$ in G. Let C be the set of connected components of $G[Y \setminus X]$. Then we have

$$\mathrm{dp}\,(X, Y) = \binom{|X|}{2} - m_X + |X| \cdot |Y \setminus X| - m_{X,Y\setminus X} + \sum_{G[Y']\in C} \min_{(X',Y')\in\mathcal{S}} \mathrm{dp}\left(X', Y'\right).$$

The cardinality of Y' is less than $|Y|$ since $X \neq \emptyset$ and as blocks are processed in increasing cardinality of Y, the value for $\mathrm{dp}\,(X', Y')$ has been calculated when it is needed for $\mathrm{dp}\,(X, Y)$.

The running time required to compute $\mathrm{dp}\,(X, Y)$ is up to a polynomial factor in k proportional to the number of sets $(X', Y') \in \mathcal{S}$, which is $\mathcal{O}(|\mathcal{S}|)$. Thus the total running time of the dynamic programming procedure is up to polynomial factor in k proportional to $\mathcal{O}(|\mathcal{S}|^2)$, and hence (1) can be computed in time $2^{\mathcal{O}(\sqrt{k}\log k)}$. This concludes Step C and the proof of Theorem 1.

4 Completion to threshold graphs

In this section we give an algorithm with running time $2^{\mathcal{O}(\sqrt{k}\log k)}n^{\mathcal{O}(1)}$ for Threshold Completion, which is the special case of $\mathcal{F}$-Completion with $\mathcal{F} = \{2K_2, C_4, P_4\}$. We aim to show the following:

Theorem 2. Threshold Completion *is solvable in time* $2^{\mathcal{O}(\sqrt{k}\log k)} + \mathcal{O}(kn^4)$*.*

The proof of Theorem 2 is a combination of the following known techniques: the kernelization algorithm by Guo [13], the chromatic coding technique of Alon et al. [1], also used in the subexponential algorithm of Ghosh et al. [11] for split graphs, and the algorithm of Fomin and Villanger for chain completion [9].

For the kernelization part we use the following result from Guo [13]. Guo stated and proved it for the complement problem Threshold Edge Deletion, but since the set of forbidden subgraphs $\mathcal{F} = \{2K_2, C_4, P_4\}$ is self-complementary, the deletion and completion problems are equivalent.

Proposition 4.1 ([13])**.** Threshold Completion *admits a kernel with* $\mathcal{O}(k^3)$ *vertices. The running time of the kernelization algorithm is* $\mathcal{O}(kn^4)$*.*

Universal sets. We start with describing the *chromatic coding* technique by Alon et al. [1]. Let χ be a coloring (not necessarily proper) of the vertex set of a graph $G = (V, E)$ into t colors. We call an edge $e \in E$ *monochromatic* if its endpoints have the same color, and we call a set of edges $F \subseteq E$ *colorful* if no edge in F is monochromatic.

Definition 4.2. A *universal* (n, k, t)*-coloring family* is a family $\mathfrak{F}$ of functions from $[n]$ to $[t]$ such that for any graph G on at most n vertices and k edges, there is an $f \in \mathfrak{F}$ such that f is a proper coloring of G, i.e., $E(G)$ is colorful.

Proposition 4.3 ([1])**.** *For any* $n > 10k^2$*, there exists an explicit universal* $(n, k, \mathcal{O}(\sqrt{k}))$*-coloring family* $\mathfrak{F}$ *of size* $|\mathfrak{F}| \leq 2^{\mathcal{O}(\sqrt{k}\log k)}\log n$*.*

Note that by *explicit* we mean here that family $\mathfrak{F}$ can be constructed in $2^{\mathcal{O}(\sqrt{k}\log k)} \cdot n^{\mathcal{O}(1)}$ time.

Split, threshold and chain graphs. Here we give some known facts about split graphs, threshold graphs and chain graphs which we will use to obtain the main result.

Definition 4.4 ([11])**.** Given a graph $G = (V, E)$, a partition of the vertex set into sets C and I is called a *split partition* of G if C is a clique and I is an independent set.

We denote by (C, I) a split partition of a graph.

Definition 4.5 (Split graph)**.** A graph is a split graph if it admits a split partitioning.

Proposition 4.6 ([11])**.** *A split graph on* n *vertices has at most* $n+1$ *split partitions and these partitions can be enumerated in polynomial time.*

Definition 4.7. A *chain graph* is a bipartite graph $G = (A, B, E)$ where the neighborhoods of the vertices are nested, i.e., there is an ordering of the vertices in A, $a_1, a_2, \ldots, a_{n_1}$, such that for each $i < n_1$ we have that $N(a_i) \subseteq N(a_{i+1})$, where $n_1 = |A|$.

We will use the following result, which is often used as an alternative definition of threshold graphs.

Proposition 4.8 ([20])**.** *A graph G is a threshold graph if and only if G has a split partition (C, I) and the neighborhoods of the vertices of I are nested.*

Thus, the class of threshold graphs is a subclass of split graphs and by Proposition 4.6, threshold graphs on n vertices have at most $n+1$ split partitions.

The algorithm. We now proceed to the details of the algorithm which solves the problem in the time stated in the theorem. Fomin and Villanger [9] showed that the following problem is solvable in subexponential time:

CHAIN COMPLETION	**Parameter:** k
Input: A bipartite graph $G = (A, B, E)$ and integer k.	
Question: Is there a set of edges $S \subseteq [V]^2$ of size at most k such that $(A, B, E \cup S)$ is a chain graph?	

Note that in the CHAIN COMPLETION problem, the resulting chain graph must have the same bipartition as the input graph. Thus, formally CHAIN COMPLETION is not an $\mathcal{F}$-COMPLETION problem according to our definition.

Proposition 4.9 ([9])**.** CHAIN COMPLETION *is solvable in $2^{\mathcal{O}(\sqrt{k}\log k)} + \mathcal{O}(k^2nm)$ time.*

With that result we have what we need to give an algorithm for THRESHOLD COMPLETION and thus prove Theorem 2.

Proof of Theorem 2. We start by using Proposition 4.1 to obtain in time $\mathcal{O}(kn^4)$ a kernel with $\mathcal{O}(k^3)$ vertices. From now on, we assume that the input graph G has $n = \mathcal{O}(k^3)$ vertices.

If (G, k) is a ***yes*** instance of THRESHOLD COMPLETION, then there is an edge set S of size at most k such that $G + S$ is a threshold graph. Without loss of generality, we can assume that $n > 10k^2$. By Proposition 4.3, one can construct in $2^{\mathcal{O}(\sqrt{k}\log k)} \cdot n^{\mathcal{O}(1)} = 2^{\mathcal{O}(\sqrt{k}\log k)}$ time an explicit universal $(n, k, \mathcal{O}(\sqrt{k}))$-coloring family $\mathfrak{F}$ of size $|\mathfrak{F}| \le 2^{\mathcal{O}(\sqrt{k}\log k)} \log n = 2^{\mathcal{O}(\sqrt{k}\log k)}$. Since $|S| \le k$, there is a vertex coloring $f \in \mathfrak{F}$ such that S is colorful.

We iterate through all the colorings $f \in \mathfrak{F}$. Let us examine one coloring $f \in \mathfrak{F}$, and let $V_1, V_2, \ldots, V_t$ be the partitioning of $V(G)$ according f, where $t = \mathcal{O}(\sqrt{k})$. Then each $G[V_i]$ is a threshold graph. By Proposition 4.6, $G+S$ has $\mathcal{O}(k^3)$ split partitions. Each such split partition of $G+S$ induces a split partition of $G[V_i]$, $i \in \{1, \ldots, t\}$. Again by Proposition 4.6, each $G[V_i]$ also has $\mathcal{O}(k^3)$ split partitions. We use brute-force to generate the set of $\mathcal{O}((k^3)^t) = 2^{\mathcal{O}(\sqrt{k}\log k)}$ partitions of G, and the set of generated partitions contains all split partitions of $G + S$. By Proposition 4.8, if (G, k) is a ***yes*** instance and f is colorful, then for at least one of the split partitions (C, I) of $G + S$ the neighborhoods of I are nested. To check if a split partition can be turned into a nested partition, we use Proposition 4.9.

To summarize, we perform the following steps.

Step A. Kernelization. Apply Proposition 4.1 to obtain in time $\mathcal{O}(kn^4)$ a kernel with $\mathcal{O}(k^3)$ vertices. From now on we assume that the number of vertices n in G is $\mathcal{O}(k^3)$.

Step B. Generating universal families. If necessary, we add a set of isolated vertices to G to guarantee that $n > 10k^2$. We apply Proposition 4.3 to construct a universal $(n, k, \mathcal{O}(\sqrt{k}))$-coloring family $\mathfrak{F}$ of size $2^{\mathcal{O}(\sqrt{k}\log k)}$. For each generated coloring f and the corresponding vertex partition $V_1, V_2, \ldots, V_t$, $t = \mathcal{O}(\sqrt{k})$, we perform the steps that follow.

Step C. Generating split partitions. We generate a set of partitions $\mathcal{C}$ of $V(G)$ as follows. Each partition $(C, I) \in \mathcal{C}$ is of the following form. For $i \in \{1, \ldots, t\}$, let $\mathcal{C}_i$, $|\mathcal{C}_i| = \mathcal{O}(k^3)$, be the set of split partitions of $G[V_i]$. Then for each $i \in \{1, \ldots, t\}$, $(C \cap V_i, I \cap V_i) \in \mathcal{C}_i$. In other words, every partition of $\mathcal{C}$ induces a split partition of $G[V_i]$. The time required to generate all partitions from $\mathcal{C}$ is $\mathcal{O}((k^3)^t) = 2^{\mathcal{O}(\sqrt{k}\log k)}$. We also perform a sanity check by excluding from $\mathcal{C}$ all pairs (C, I), where I is not an independent set. We perform the next step with each pair $(C, I) \in \mathcal{C}$.

Step D. Computing nested split partitions. For pair $(C, I) \in \mathcal{C}$, such that I is an independent set in G, we first compute the number of edges c needed to turn C into a clique, i.e., $c = \binom{|C|}{2} - m_C$. Finally, we use Proposition 4.9 to check if the neighborhood of I in C can be made nested by adding at most $k - c$ edges.

From the discussions above, if (G, k) is a ***yes*** instance of the problem, the solution will be found after completing the algorithm. Otherwise, we conclude that (G, k) is a ***no*** instance. The running time to perform Step A is $\mathcal{O}(kn^4)$ and Step B is done in $2^{\mathcal{O}(\sqrt{k}\log k)}$. For every $f \in \mathfrak{F}$, in Step C we generate $2^{\mathcal{O}(\sqrt{k}\log k)}$ partitions. The total number of times Step C is called is $|\mathfrak{F}|$ and the total number of partitions generated is $|\mathfrak{F}| \cdot 2^{\mathcal{O}(\sqrt{k}\log k)} = 2^{\mathcal{O}(\sqrt{k}\log k)}$. In Step D, we run the algorithm with running time $2^{\mathcal{O}(\sqrt{k}\log k)}$ on each of the $2^{\mathcal{O}(\sqrt{k}\log k)}$ partitions, resulting in total running time $2^{\mathcal{O}(\sqrt{k}\log k)} + \mathcal{O}(kn^4)$. □

5 Completion to pseudosplit graphs

In this section we show that Pseudosplit Completion, or $\mathcal{F}$-Completion for $\mathcal{F} = \{2K_2, C_4\}$, can be solved by first applying a polynomial-time and parameter-preserving preprocessing routine, and then using the subexponential time algorithm of Ghosh et al. [11] for Split Completion.

The crucial property of pseudosplit graphs that will be of use is the following characterization:

Proposition 5.1 ([19])**.** *A graph $G = (V, E)$ is pseudosplit if and only if one of the following holds*

- *G is a split graph, or*
- *V can be partitioned into C, I, X such that $G[C \cup I]$ is a split graph with C being a clique and I being an independent set, $G[X] \cong C_5$, and moreover, there is no edge between X and I and every edge is present between X and C.*

In other words, a pseudosplit graph is either a split graph, or a split graph containing one induced C_5 which is completely non-adjacent to the independent set of the split graph, and completely adjacent to the clique set of the split graph. We call a graph which falls into the latter category a *proper pseudosplit graph.*

In order to ease the argumentation regarding minimal completions, we call a split partition (C, I) *I-maximal* if there is no vertex $v \in C$ such that $(C \setminus \{v\}, I \cup \{v\})$ is a split partition. Our algorithm uses the subexponential algorithm of Ghosh et al. [11] for Split Completion as a subroutine. We therefore need the following result:

Proposition 5.2 ([11])**.** *The problem* Split Completion *is solvable in time* $2^{\mathcal{O}(\sqrt{k}\log k)}n^{\mathcal{O}(1)}$.

Formally, in this section we prove the following theorem:

Theorem 3. Pseudosplit Completion *is solvable in time* $2^{\mathcal{O}(\sqrt{k}\log k)}n^{\mathcal{O}(1)}$.

Algorithm 1 Algorithm solving PSEUDOSPLIT COMPLETION.

1. Use the algorithm from Proposition 5.2 to check in time $2^{\mathcal{O}(\sqrt{k}\log k)}n^{\mathcal{O}(1)}$ if (G, k) is a ***yes*** instance of SPLIT COMPLETION. If (G, k) is a ***yes*** instance of SPLIT COMPLETION, then return that (G, k) is a ***yes*** instance of PSEUDOSPLIT COMPLETION. Otherwise we complete to a proper pseudosplit graph.
2. For each $X = \{x_1, x_2, \ldots, x_5\} \subseteq V(G)$ such that there is a supergraph $G_X \supseteq G[X]$ and $G_X \cong C_5$, we construct a new graph G' from G as follows
   (a) Add all the possible edges between vertices of X, so that X becomes a clique.
   (b) Add a set A of $k + 2$ vertices to G.
   (c) Add every possible edge between A and $N_G[X]$.
3. Let $k' = k + |E(G[X])| - 5$. Use Proposition 5.2 to check if (G', k') is a ***yes*** instance of SPLIT COMPLETION. If (G', k') is a ***yes*** instance of SPLIT COMPLETION, then return that (G, k) is a ***yes*** instance of PSEUDOSPLIT COMPLETION.
4. If for no set X the answer ***yes*** was returned, then return ***no***.

The algorithm whose existence is asserted in Theorem 3 is given as Algorithm 1. We now proceed to prove that this algorithm is correct, and that its running time on input (G, k) is $2^{\mathcal{O}(\sqrt{k}\log k)}n^{\mathcal{O}(1)}$. In the following we adopt the notation from Algorithm 1.

As in the algorithm, we denote by X the set of five vertices which will be used as the set inducing a C_5 (we try all possible subsets; note that their number is bounded by $\mathcal{O}(n^5)$). Note here that since $G[X]$ admits a supergraph isomorphic to a C_5, it follows that $|E(G[X])| \le 5$ and, consequently, $k' \le k$. Similarly, by A we denote the set of $k + 2$ vertices we add that are adjacent only to $N_G[X]$. Intuitively, this set will be used to force that in any minimal split completion of size at most k it holds that $N_G[X] \subseteq C$. From now on G' is the graph as in the algorithm, that is, G' is constructed from G by making X into a clique, adding vertices of A and all the possible edges between A and $N_G[X]$.

The following lemma will be crucial in the proof of algorithm's correctness.

Lemma 5.3. *Assume that S is a minimal split completion of G' of size at most k', and let (C, I) be an I-maximal split partition of $G' + S$. Then:*

(i) $N_G[X] \subseteq C$,

(ii) $A \subseteq I$,

(iii) no edge of S has an endpoint in A,

(iv) $C \setminus X$ is fully adjacent to X in $G' + S$, and

(v) $I \setminus A$ is fully non-adjacent to X in $G' + S$.

Proof. *(i)* Aiming towards a contradiction, suppose that some $v \in N_G[X]$ is in I. Since $A \subseteq N(v)$, we must then have that $A \subseteq C$. However, since A is stable in G, this demands adding at least $\binom{k+2}{2} > k'$ edges.

(ii) Aiming towards a contradiction, assume that $A \cap C \neq \emptyset$. Since $N_G(A) \subseteq C$ and A is stable in G, it follows that $G' + S'$, where S' is S with all the edges incident to A removed, is also a

split graph with partition (C', I'), where $C' = C \setminus A$ and $I' = I \cup (A \cap C)$. Since $S' \subseteq S$, we have that either $|S'| < |S|$ which is a contradiction with minimality of S, or that $S' = S$ and we obtain a contradiction with the assumption that partition (C, I) was I-maximal.

(iii) Suppose that there is an edge $e \in S$ incident to a vertex of A. Since $A \subseteq I$, we infer that $S \setminus \{e\}$ is still a split completion, which contradicts the minimality of S.

(iv) C is a clique in $G' + S$ and $X \subseteq C$, so this holds trivially.

(v) Suppose for a contradiction that some $v^i \in I \setminus A$ is adjacent to some $v^x \in X$. Since $N_G[X] \subseteq C$, we have that $v^i v^x \in S$. But then $S \setminus \{v^i v^x\}$ is also a split completion, and we have a contradiction with the minimality of S. □

The correctness of the algorithm is implied by the following lemma:

Lemma 5.4. *The instance (G, k) is a* ***yes*** *instance of* Pseudosplit Completion *if and only if Algorithm 1 returns* ***yes*** *on input (G, k).*

Proof. From left to right, let (G, k) be a ***yes*** instance for Pseudosplit Completion. We immediately observe that (G, k) is a ***yes*** instance for Split Completion if and only if our algorithm returns ***yes*** in the first test. We therefore assume that G has to be completed to a proper pseudosplit graph.

Let S_0 be a completion set with $|S_0| \leq k$ such that $G_0 = G + S_0$ is a proper pseudosplit graph. Let (C, I, X) be the pseudosplit partition of $G + S_0$; hence $G_0[X]$ is isomorphic to a C_5. We claim that the algorithm will return ***yes*** when considering the set X in the second point; let then G' be the graph constructed in the algorithm for the set X. Let S be equal to S_0 with all the edges of $G_0[X]$ that were not present in $G[X]$ removed; note that $|S| = |S_0| + |E(G[X])| - 5 \leq k'$. We claim that $G' + S$ is a split graph with split partition $(C \cup X, I \cup A)$. Indeed, since $G'[X]$ is a clique, X is fully adjacent to C in $G_0 \subseteq G' + S$, and C is a clique in $G_0 \subseteq G' + S$, then $C \cup X$ is a clique in $G' + S$. On the other hand, $I \cup A$ is independent in G' and all the edges of S have at least one endpoint belonging to $C \cup X$, so $I \cup A$ remains independent in $G' + S$. As a result $G' + S$ is a split graph, and so the algorithm will return ***yes*** after the application of Proposition 5.2 in the third point.

From right to left, assume that Algorithm 1 returns ***yes*** on input (G, k). If it returned ***yes*** already on the first test, then G may be completed into a split graph by adding at most k edges, so in particular (G, k) is a ***yes*** instance of Pseudosplit Completion. From now on we assume that the algorithm returned ***yes*** in the third point. More precisely, for some set X the application of Proposition 5.2 has found a minimal completion set S of size at most k' such that $G' + S$ is a split graph, with I-maximal split partition (C, I).

By Lemma 5.3 we have that *(i)* $N_G[X] \subseteq C$, *(ii)* $A \subseteq I$, *(iii)* no edge of S has an endpoint in A, *(iv)* $C \setminus X$ is fully adjacent to X in $G' + S$, and *(v)* $I \setminus A$ is fully non-adjacent to X in $G' + S$. By the choice of X, there exists a supergraph G_X of $G[X]$ such that $G_X \cong C_5$. Let now S_0 be equal to S with all the edges of G_X that were not present in $G[X]$ included. Observe that $|S_0| \leq k$ and that by *(iii)* S_0 contains only edges incident to vertices of G. Consider now the partition $(C \setminus X, I \setminus A, X)$ of $V(G + S_0)$. Since (C, I) was a split partition of $G' + S$, it follows that $C \setminus X$ is a clique in $G + S_0$ and $I \setminus A$ is an independent set in $G + S_0$. Moreover, from *(iv)* and *(v)* it follows that X is fully adjacent to $C \setminus X$ in $G + S_0$ and fully non-adjacent to $I \setminus A$ in $G + S_0$. Finally, the graph induced by X in $G + S_0$ is $G_X \cong C_5$. By Lemma 5.1 we infer that $G + S_0$ is a pseudosplit graph, and so the instance (G, k) is a ***yes*** instance of Pseudosplit Completion. □

As for the time complexity of the algorithm, we try sets of five vertices for X, which is $\mathcal{O}(n^5)$ tries. For each such guess, we construct the graph G', which has $n + k + 2$ vertices. Since

$k' \le k$, by Proposition 5.2 solving SPLIT COMPLETION requires time $2^{\mathcal{O}(\sqrt{k}\log k)} n^{\mathcal{O}(1)}$, both in the first and the third point of the algorithm. Thus the total running time of Algorithm 1 is $2^{\mathcal{O}(\sqrt{k}\log k)} n^{\mathcal{O}(1)}$.

6 Lower bounds

In this section we show that if $\mathcal{F}$ is any singleton subset of $\{2K_2, C_4, P_4\}$, then $\mathcal{F}$-COMPLETION is not solvable in subexponential time unless ETH fails. It also follows from the proof for P_4-FREE COMPLETION that $\mathcal{F}$-COMPLETION is not solvable in subexponential time when $\mathcal{F} = \{2K_2, P_4\}$. In other words, CO-TRIVIALLY PERFECT COMPLETION is not solvable in subexponential time unless ETH fails. Summarizing, we show that $\mathcal{F}$-COMPLETION is not solvable in subexponential time unless ETH fails for:

- $\mathcal{F} = \{2K_2\}$,
- $\mathcal{F} = \{C_4\}$,
- $\mathcal{F} = \{P_4\}$,
- and $\mathcal{F} = \{2K_2, P_4\}$.

Let us observe that the $\mathcal{F} = \{2K_2, P_4\}$ problem is polynomial-time equivalent to TRIVIALLY PERFECT EDGE DELETION.

Throughout this section we will reduce to the above problems from 3SAT. Therefore, we will assume that the input formula φ is in 3-CNF, that is, it is a conjunction of a number of clauses, where each clause is a disjunction of at most 3 literals. By applying standard regularizing preprocessing for φ (see for instance [8, Lemma 13]) we will also assume that each clause of φ contains exactly 3 literals, and the variables appearing in these literals are pairwise different.

If φ is a 3SAT instance, we denote by $\mathcal{V}(\varphi)$ the variables in φ and by $\mathcal{C}(\varphi)$ the clauses. We assume we have an ordering $c_1, c_2, \ldots, c_m$ for the clauses in $\mathcal{C}(\varphi)$ and the same for the variables, $x_1, x_2, \ldots, x_n$. For simplicity, we also assume the literals in each clause are ordered internally by the variable ordering.

We restate here the Exponential Time Hypothesis, as that will be the crucial assumption for proving that the problems mentioned above do not admit subexponential time algorithms.

Exponential Time Hypothesis (ETH). There exists a positive real number s such that 3SAT with n variables and m clauses cannot be solved in time $2^{sn}(n+m)^{\mathcal{O}(1)}$.

By the Sparsification Lemma of Impagliazzo, Paturi and Zane [14], unless ETH fails, 3SAT also cannot be solved in time $2^{o(n+m)}(n+m)^{\mathcal{O}(1)}$.

For each considered problem we present a linear reduction from 3SAT, that is, a reduction which constructs an instance whose parameter is bounded linearly in the size of the input formula. Pipelining such a reduction with the assumed subexponential parameterized algorithm for the problem would give a subexponential algorithm for 3SAT, contradicting ETH. A reader familiar with the results of Cai [5] will find similarities between gadgets used there and gadgets introduced in this section, especially when C_4s are concerned. Since the goal of the reductions presented in [5] is showing hardness of polynomial kernelization rather than non-existence of subexponential algorithms, in [5] it was not necessary to make the reduction linear. Instead, one needed to start with more carefully chosen variants of SAT. We expect that some of the constructions of Cai [5] can be indeed reused in our reductions; however, to make our study complete and self-contained we choose to present our own, independently obtained proofs.

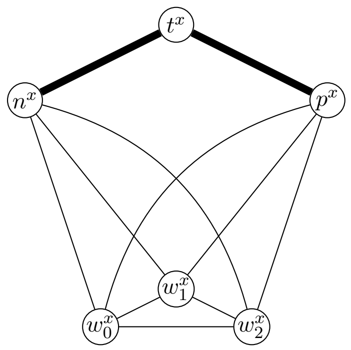
6.1 $2K_2$-free completion is not solvable in subexponential time

For $\mathcal{F} = \{2K_2\}$, we refer to $\mathcal{F}$-Completion as to $2K_2$-Free Completion. We show the following theorem.

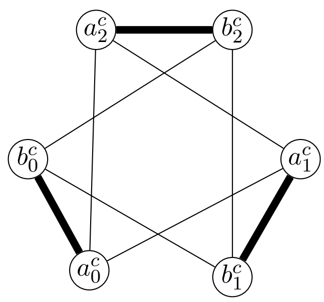
Theorem 4. *The problem $2K_2$-Free Completion is not solvable in $2^{o(k)} n^{\mathcal{O}(1)}$ time unless the Exponential Time Hypothesis (ETH) fails.*

For the proof, however, instead of working directly on this problem, we find it more convenient to show the hardness of the (polynomially) equivalent problem C_4-Free Edge Deletion.

Construction. We reduce from 3Sat and the gadgets can be seen in Figure 5. Let φ be an instance for 3Sat. We construct the graph G_φ for C_4-Free Edge Deletion. For every variable $x \in \mathcal{V}(\varphi)$, we construct a variable gadget graph G^x. The graph G^x consists of six vertices w_0^x, w_1^x, w_2^x, n^x (for *negative*), p^x (for *positive*), and t^x. The three vertices w_0^x, w_1^x and w_2^x will induce a triangle whereas n^x and p^x are adjacent to the vertices in the triangle and to t^x. We can observe that the four vertices n^x, t^x, p^x, w_i^x induces a C_4 for $i = 0, 1, 2$, and that no other induced C_4 occurs in the gadget (see Figure 5a). It can also be observed that by removing either one of the edges $n^x t^x$ and $p^x t^x$, the gadget becomes C_4-free. We will refer to the edge $t^x p^x$ as the *true edge* and to $t^x n^x$ as the *false edge*. These edges are the thick edges in Figure 5a. This concludes the variable gadget construction.



(a) The variable gadget G^x for a variable x with three occurrences of C_4. The edge $t^x p^x$ is the true edge and the edge $t^x n^x$ is the false edge. All C_4s of G^x can be eliminated by removing the true or the false edge.



(b) The clause gadget G^c for a clause c has three occurrences of C_4, which can be eliminated by removing two of the variable-edges, the thick edges in the figure.

Figure 5: Variable (left) and clause (right) gadgets used in the reduction.

For every clause $c \in \mathcal{C}(\varphi)$, we construct a clause gadget graph G^c as follows. The graph G^c consists of two triangles, a_0^c, a_1^c, a_2^c and b_0^c, b_1^c, b_2^c. We also add the edges $a_0^c b_0^c$, $a_1^c b_1^c$, and $a_2^c b_2^c$. These three latter edges will correspond to the variables contained in c and we refer to them as *variable-edges* (the thick edges in Figure 5b). No more edges are added. The clause gadget can be seen in Figure 5b. Observe that there are exactly three induced C_4s in G^c, all of the form $a_i^c, a_{i+1}^c, b_{i+1}^c, b_i^c$ for $i = 0, 1, 2$, where the indices behave cyclically modulo 3. Moreover, removing any two edges of the form $a_i^c b_i^c$ for $i = 0, 1, 2$ eliminates all the induced C_4s contained in G^c.

To conclude the construction, we give the connections between variable gadgets and clause gadgets that encode literals in the clauses (see Figure 6). If a variable x appears in a non-negated form as the ith (for $i = 0, 1, 2$) variable in a clause c, we add the edges $t^x a_i^c$ and $p^x b_i^c$. If it appears in a negated form, we add the edges $t^x a_i^c$ and $n^x b_i^c$. The connections can be seen

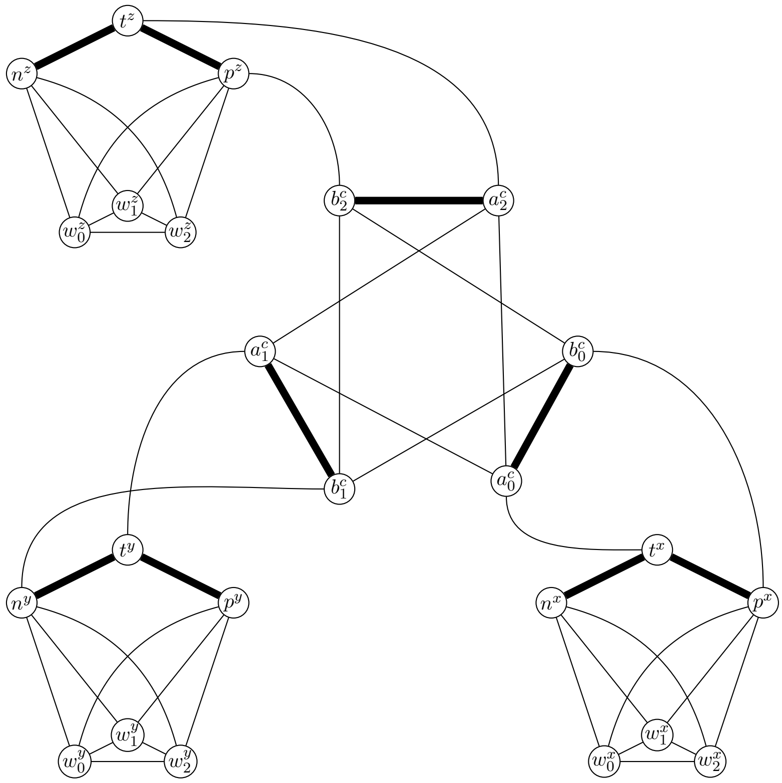


Figure 6: The connections for a clause $c = x \vee \neg y \vee z$. For the negated variable, $\neg y$, we connect the clause gadget to n^y and t^y, whereas for the variables in the non-negated form we have the clause connected to the t and p vertices. Observe that a budget of five is sufficient and necessary for eliminating all occurrences of C_4 in the depicted subgraph.

in Figure 6. Observe that we get exactly one extra induced C_4 in the connection, and that this can be eliminated by removing either one of the thick edges.

This concludes the construction. We have now obtained a graph G_φ constructed from an instance φ of 3SAT. We let $k_\varphi = |\mathcal{V}(\varphi)| + 2|\mathcal{C}(\varphi)|$ be the allowed (and necessary) budget, and the instance of C_4-FREE EDGE DELETION is then (G_φ, k_φ).

We now proceed to prove the following lemma, which will give the result.

Lemma 6.1. *A given 3-CNF formula φ has a satisfying assignment if and only if (G_φ, k_φ) is a **yes** instance of C_4-FREE EDGE DELETION.*

Proof. Let φ be satisfiable and G_φ and k_φ be as above. We show that (G_φ, k_φ) is a ***yes*** instance for C_4-FREE EDGE DELETION. Let $\alpha\colon \mathcal{V}(\varphi) \to \{\texttt{true}, \texttt{false}\}$ be a satisfying assignment for φ. For every variable $x \in \mathcal{V}(\varphi)$, if $\alpha(x) = \texttt{true}$, we remove the edge corresponding to true, i.e. the edge $t^x p^x$, otherwise we remove the edge corresponding to false, i.e., the edge $t^x n^x$. Every clause $c \in \mathcal{C}$ is satisfied by α; we pick an arbitrary variable x whose literal satisfies c and remove two edges corresponding to the two other literals. If a clause is satisfied by more than one literal, we pick any of the corresponding variables.

For every clause we deleted exactly two edges and for every variable exactly one edge. Thus the total number of edges removed is $2|\mathcal{C}(\varphi)| + |\mathcal{V}(\varphi)| = k_\varphi$. We argue now that the remaining graph G'_φ is C_4-free. Since variables appearing in clauses are pairwise different, it can be easily observed that every induced cycle of length four in G_φ is either

- entirely contained in some clause gadget, or

- entirely contained in some variable gadget, or
- is of form $t^x\gamma^x b^c_i a^c_i$, where x is the ith variable of clause c, and $\gamma \in \{n,p\}$ denotes whether the literal in c that corresponds to x is negated or non-negated.

By the construction of G'_φ, we destroyed all induced 4-cycles of the first two types. Consider a 4-cycle $t^x p^x b^c_i a^c_i$ of the third type, where x appears positively in clause c. In the case when the literal of variable x was not chosen to satisfy c, we have deleted the edge $a^c_i b^c_i$ and so this 4-cycle is removed. Otherwise we have that $\alpha(x) = \texttt{true}$, and we have deleted the edge $t^x p^x$, thus also removing the considered 4-cycle. The case of a 4-cycle of the form $t^x n^x b^c_i a^c_i$ is symmetric.

Concluding, all the induced 4-cycles that were contained in G_φ are removed in G'_φ. Since vertices in pairs (a^c_i, b^c_i) and (γ^x, t^x) for $\gamma \in \{n,p\}$ do not have common neighbours, it follows that no new C_4 could be created when obtaining G'_φ from G_φ by removing the edges. We infer that G'_φ is indeed C_4-free.

We proceed with the opposite direction. Let S be an edge set of G_φ of size at most k_φ such that $G - S$ is C_4-free. By the definition of the budget k_φ and the observation that every variable gadget needs at least one edge to be in S and every clause gadget needs at least two edges to be in S (note here that the edge sets of clause and variable gadgets are pairwise disjoint), we have that S contains *exactly* one edge from each variable gadget, *exactly* two edges from each clause gadget, and no other edges.

We construct an assignment $\alpha \colon \mathcal{V}(\varphi) \to \{\texttt{true}, \texttt{false}\}$ for the formula φ as follows. For a variable $x \in \mathcal{V}(\varphi)$, put $\alpha(x) = \texttt{false}$ if the false edge $t^x n^x$ of G^x is in S, put $\alpha(x) = \texttt{true}$ if the true edge $t^x p^x$ of G^x is in S, and put an arbitrary value for $\alpha(x)$ otherwise. We claim that the assignment α satisfies φ.

Suppose for a contradiction that a clause $c \in \mathcal{C}$ is not satisfied. Since exactly two edges in the clause gadget G^c are in S, there is a variable x appearing in c such that the corresponding variable-edge of G^c is not in S. If $\alpha(x) = \texttt{true}$, then because c is not satisfied, we have that $\neg x \in c$. By the definition of α we have that the false edge of G^x does not belong to S. Then in G_φ, the false edge of G^x and the variable-edge of G^c corresponding to x form an induced C_4 that is not destroyed by S, a contradiction. The case $\alpha(x) = \texttt{false}$ is symmetric. This concludes the proof of the lemma. □

Finally, the proof of Theorem 4 follows from Lemma 6.1: pipelining the presented reduction with an algorithm for C_4-Free Edge Deletion working in $2^{o(k)} n^{\mathcal{O}(1)}$ time would give an algorithm for 3Sat working in $2^{o(n+m)}(n+m)^{\mathcal{O}(1)}$ time, which contradicts ETH by the results of Impagliazzo, Paturi and Zane [14].

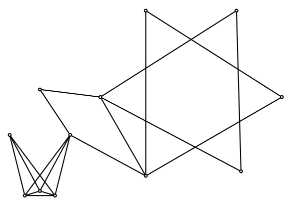


Figure 7: Connections

6.2 C_4-free completion is not solvable in subexponential time

Every instance of $\mathcal{F}$-Completion we have considered which turned out to be solvable in subexponential time so far has had C_4 in $\mathcal{F}$ together with some other graphs: trivially perfect graphs is the class excluding C_4 and P_4, threshold graphs is the class excluding $2K_2$, P_4 and

(a) Before completion (b) True (c) False (d) Differing orientations

Figure 8: The variable gadget G_x.

C_4, and pseudosplit graphs is the class excluding $2K_2$ and C_4. It is natural to ask if the C_4 is the "reason" for the existence of subexponential algorithms.

In this section, we show that excluding only C_4 does not make $\mathcal{F}$-Completion solvable in subexponential time. For $\mathcal{F} = \{C_4\}$, we refer to $\mathcal{F}$-Completion as C_4-Free Completion.

Theorem 5. *The problem C_4-Free Completion is not solvable in $2^{o(k)}n^{\mathcal{O}(1)}$ time unless the Exponential Time Hypothesis (ETH) fails.*

We now show that the problem is not solvable in subexponential time. To this end, we reduce from 3Sat, and similarly as before we start with a formula where each clause contains exactly three literals corresponding to pairwise different variables. By duplicating clauses if necessary, we also assume that each variable appears in at least two clauses.

We again need two types of gadgets, one gadget to emulate variables in the formula and one type to emulate clauses. Let φ be the input 3Sat instance and denote by $\mathcal{V}(\varphi)$ the variables in φ and by $\mathcal{C}(\varphi)$ the clauses. We construct the graph G_φ as follows:

For each variable $x \in \mathcal{V}(\varphi)$ we construct a variable gadget graph G^x as depicted in Figure 9. Let p_x be the number of clauses x occurs in; by our assumption we have that $p_x \geq 2$. The graph G^x consists of a "tape" of $4p_x$ squares arranged on a cycle, with additional vertices attached to the sides of the tape. The intuition is that every fourth square in G^x is reserved for a clause x occurs in. Formally, the vertex set of G^x consists of

$$V(G^x) = \bigcup_{0 \leq i < 4p_x} \{u_i^x, t_i^x, b_i^x, d_i^x\},$$

and the edge set of

$$E(G^x) = \bigcup_{0 \leq i < 4p_x} \{u_i^x t_i^x, u_i^x t_{i+1}^x, t_i^x u_{i+1}^x, t_i^x t_{i+1}^x, t_i^x b_i^x, b_i^x b_{i+1}^x, b_i^x d_{i+1}^x, b_i^x d_i^x, d_i^x b_{i+1}^x\},$$

where the indices behave cyclically modulo $4p_x$. The construction is visualized in Figure 9.

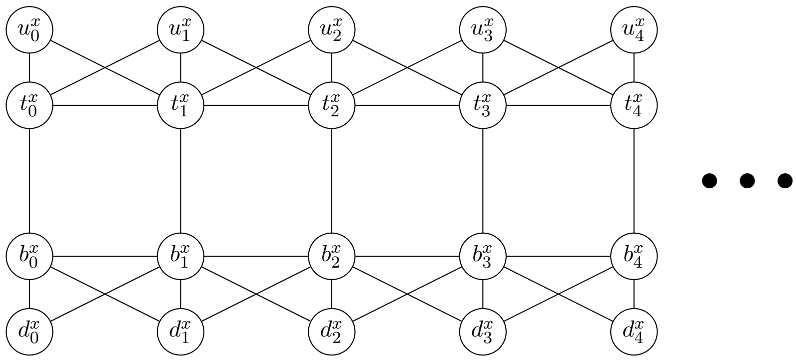


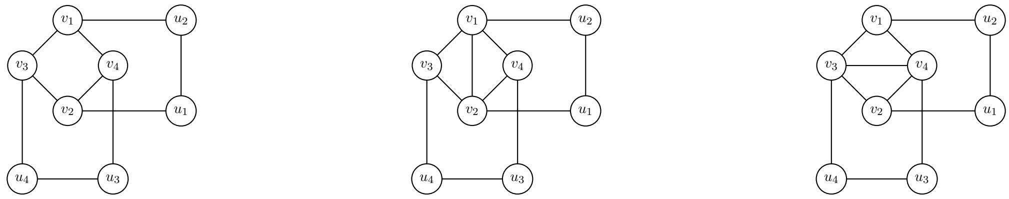
Figure 9: Variable gadget G_x.

Claim 6.2. *The minimum number of edges required to add to G^x to make it C_4-free is $4p_x$. Moreover, there are exactly two ways of eliminating all C_4s with $4p_x$ edges, namely adding an edge on the diagonal for each square. Furthermore, if we add one edge to eliminate some cycle, all the rest must have the same orientation, i.e., either all added edges are of the form $t^x_i b^x_{i+1}$ or of the form $t^x_{i+1} b^x_i$. See Figure 8.*

Proof of claim. A gadget G^x contains $4p_x$ induced C_4, and no two of them can be eliminated by adding one edge. Hence, to eliminate all C_4s in G^x, we need at least $4p_x$ edges. On the other hand, it is easy to verify that after adding $4p_x$ diagonals to C_4s of the same orientation the resulting graph does not contain any induced C_4, see Figure 8 for examples. Whenever we have two consecutive cycles with completion edges of different orientation, we create a new C_4 consisting of the two completion edges, and (depending on their orientation) either two edges incident to vertex u^x_i above their common vertex, or two edges incident to vertex d^x_i below. See Figure 8d. □

Corollary 6.3. *The minimum number of edges required to eliminate all C_4s appearing inside all the variable gadgets is $12|\mathcal{C}(\varphi)|$.*

Proof. Since each clause of $\mathcal{C}(\varphi)$ contains exactly three occurrences of variables, it follows that $\sum_{x\in\mathcal{V}(\varphi)} p_x = 3|\mathcal{C}(\varphi)|$. The constructed variable gadgets are pairwise disjoint, so by Claim 6.2 we infer that the minimum number of edges required in all the variable gadgets is equal to $\sum_{x\in\mathcal{V}(\varphi)} 4p_x = 3\cdot 4|\mathcal{C}(\varphi)| = 12|\mathcal{C}(\varphi)|$. □



(a) Before completion

(b) Variable 1 or 2 in the clause must be satisfied

(c) Variable 3 in the clause must be satisfied

Figure 10: The clause gadget

We now proceed to create the clause gadgets. For each clause $c \in \mathcal{C}(\varphi)$, we create the graph G^c as depicted in Figure 11. It consists of an induced 4-cycle $v^c_1 v^c_4 v^c_2 v^c_3$ and induced paths $v^c_2 u^c_1 u^c_2 v^c_1$ and $v^c_3 u^c_4 u^c_3 v^c_4$. We also attach a gadget consisting of k_φ internally disjoint induced paths of four vertices with endpoints in v^c_4 and u^c_4, where k_φ is the budget to be specified later. That makes it impossible to add an edge between v^c_4 and u^c_4 in any C_4-free completion with at most k_φ edges.

By the i-th *square* we mean a quadruple $(t^x_i, b^x_i, t^x_{i+1}, b^x_{i+1})$. If a clause c is the ℓ-th clause the variable x appears in, we will use the vertices of the $4(\ell-1)$-st square for connections to the gadget corresponding to c. For ease of notation let $j = 4(\ell-1)$. We also use pairs $\{v^c_1, u^c_1\}$, $\{v^c_2, u^c_2\}$, and $\{v^c_3, u^c_3\}$ of G^c for connecting to the corresponding variable gadgets. If a variable x appears in a non-negated form as the ith (for $i = 1, 2, 3$) literal of a clause c, then we add the edges $t^x_{j+1} v^c_i$ and $b^x_j u^c_i$. If it appears in a negated form, we add the edges $t^x_j v^c_i$ and $b^x_{j+1} u^c_i$. See Figure 12. This concludes the construction of G_φ. Finally, we set the budget for the instance equal to $k_\varphi = 14|\mathcal{C}(\varphi)|$.

Claim 6.4. *For each clause gadget G^c for a clause $c \in \mathcal{C}(\varphi)$, we need to add at least* two edges *between vertices of G^c to eliminate all induced C_4s in G^c. Moreover, there are exactly three*

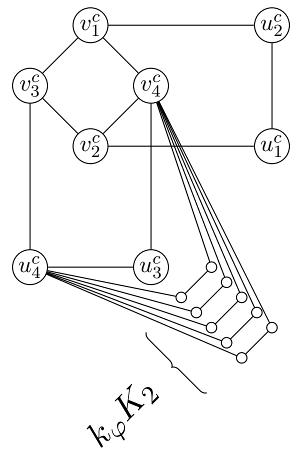


Figure 11: The clause gadget G^c. It contains one C_4, and if we add either edge $v_1^c v_2^c$ or edge $v_3^c v_4^c$, we get a new C_4 that must be destroyed by adding one more edge. The $k_\varphi K_2$ gadget makes sure we cannot add edge $v_4^c u_4^c$.

ways of adding exactly two edges to G^c so that the resulting graph does not contain any induced C_4: by adding $\{v_1^c v_2^c, v_1^c u_1^c\}$, $\{v_1^c v_2^c, v_2^c u_2^c\}$, or $\{v_3^c v_4^c, v_3^c u_3^c\}$.

Proof of claim. There is a four-cycle $v_1^c v_4^c v_2^c v_3^c$ which needs to be eliminated, either by adding the edge $v_1^c v_2^c$ (Figure 10b) or $v_3^c v_4^c$ (Figure 10c). In any case we create a new C_4, either $v_1^c u_2^c u_1^c v_2^c$ in the former case, and $v_4^c u_3^c u_4^c v_3^c$ in the latter case. In the former case we can eliminate the created C_4 by adding $v_1^c u_1^c$ or $v_2^c u_2^c$, and in the latter case we can eliminate it by adding $v_3^c u_3^c$. Note that in the latter case we cannot add $v_4^c u_4^c$, since then we would create k_φ new induced four-cycles. A direct check shows that all the three aforementioned completion sets lead to a C_4-free graph. □

Lemma 6.5. *Given a* 3Sat *instance φ, we have that (G_φ, k_φ) is a* ***yes*** *instance for C_4-*Free Completion *for $k_\varphi = 14|\mathcal{C}(\varphi)|$ if and only if φ is satisfiable.*

Proof. From right to left, suppose φ is satisfiable. Let $\alpha\colon \mathcal{V}(\varphi) \to \{\texttt{true}, \texttt{false}\}$ be a satisfying assignment for φ. For every variable $x \in \mathcal{V}(\varphi)$, if $\alpha(x) = \texttt{true}$, we add edges $t_i^x b_{i+1}^x$ to S for $i \in \{0, \ldots, 4p_x - 1\}$ and if $\alpha(x) = \texttt{false}$, we add edges $t_{i+1}^x b_i^x$ to S for $i \in \{0, \ldots, 4p_x - 1\}$.

For a clause c in $\mathcal{C}(\varphi)$, if the first literal satisfies the clause, we add the edges $v_1^c v_2^c$ and $v_1^c u_1^c$ to S. If the second literal satisfies the clause, we add $v_1^c v_2^c$ and $v_2^c u_2^c$ to S and if it is the third literal, we add $v_3^c v_4^c$ and $v_3^c u_3^c$ to S. If more than one literal satisfies the clause, we pick any. In total this makes $12|\mathcal{C}(\varphi)|$ edges added to the variable gadgets and $2|\mathcal{C}(\varphi)|$ edges added to the clause gadgets.

Suppose now for a contradiction that $G_\varphi + S$ contains a cycle L of length four. In Claims 6.2 and 6.4 it is already verified that L is not completely contained in a variable or clause gadget. Each vertex has at most one incident edge ending outside the gadget of the vertex and there are only edges between variable and clause gadgets. Thus L consist of one edge from a variable gadget and one from a clause gadget and two edges between. We can observe that L then must contain either $v_1^c u_1^c$, $v_2^c u_2^c$, or $v_3^c u_3^c$ of the clause gadget, see Figure 12. Let us assume without loss of generality that L contains the edge $v_1^c u_1^c$. By the construction of the set S this implies that the literal of the first variable x of c satisfies c. If x is non-negated in c, then we have that $\alpha(x) = \texttt{true}$ and that $v_1^c t_{j+1}^x$ and $u_1^c b_j^x$ are edges of L. To complete the cycle $t_{j+1}^x b_j^x$ must be an edge of L; however, by the definition of S we have added the edge $t_j^x b_{j+1}^x$ to S instead of $t_{j+1}^x b_j^x$, and we obtain a contradiction. The case where x is negated is symmetric.

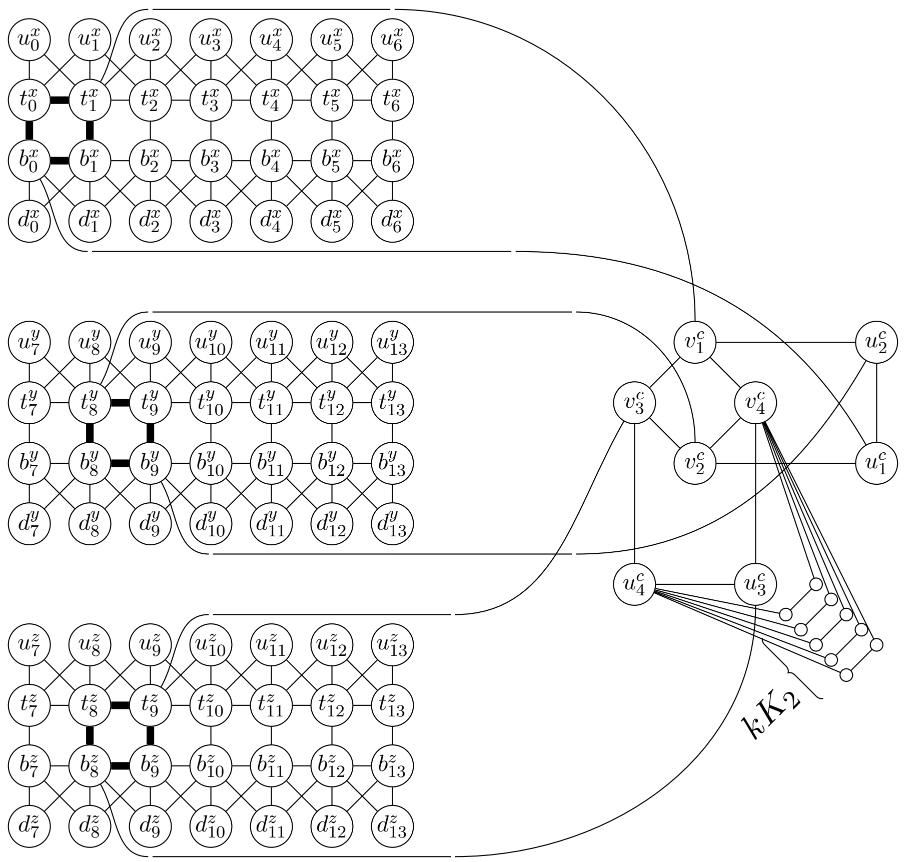


Figure 12: The connections for a clause $c = x \vee \neg y \vee z$. In this example, c is the first clause of appearance for x thus x is connected to G^c via the 0th square. For y and z, we assume that c is the third clause they appear, thus y and z use the 8th square.

From left to right, suppose (G_φ, k_φ) is a ***yes*** instance for $k_\varphi = 14|\mathcal{C}(\varphi)|$ and let S be such that $G_\varphi + S$ is C_4-free with $|S| \leq k_\varphi$. By Corollary 6.3 and Claim 6.4 we know that we need to use at least $12|\mathcal{C}(\varphi)|$ edges to fix the variable gadgets and we need to use at least $2|\mathcal{C}(\varphi)|$ edges for the clause gadgets. Since $|S| \leq k_\varphi$, we infer that $|S| = k_\varphi$, that we use at exactly $4p_x$ edges to fix each variable gadgets G^x (and that the orientation of the added edges must be the same within the gadget), that we use exactly two edges for each clause gadget G^c, and that S contains no edges other than the mentioned above.

We now define an assignment α for $\mathcal{V}(\varphi)$ and prove that it is indeed a satisfying assignment. If S contains the edge $t_0^x b_1^x$, we let $\alpha(x) = \texttt{true}$, and if S contains the edge $t_1^x b_0^x$ we let $\alpha(x) = \texttt{false}$. Let $c \in \mathcal{C}(\varphi)$ be a clause and suppose that c is not satisfied. We know by Claim 6.4 that the gadget for c contains $\{v_1^c v_2^c, v_1^c u_1^c\}$ or $\{v_1^c v_2^c, v_2^c u_2^c\}$, or $\{v_3^c v_4^c, v_3^c u_3^c\}$.

Without loss of generality assume that G^c contains $\{v_1^c v_2^c, v_1^c u_1^c\}$ and that x is the first variable in c, and it appears non-negated. Since x does not satisfy c, we infer that $\alpha(x) = \texttt{false}$. This means that $t_1^x b_0^x \in S$, and since the orientation of the added edges in the gadget G^x is the same, then also $t_{i+1}^x b_i^x \in S$. As a result, both edges $t_{i+1}^x b_i^x$ and $v_1^c u_1^c$ are present in $G_\varphi + S$. But then we have an induced four-cycle $v_1^c u_1^c b_i^x t_{i+1}^x v_1^c$, contradicting the assumption that $G_\varphi + S$ was C_4-free. The cases for y, z and negative literals are symmetric. This concludes the proof. ☐

Similarly as before, the proof of Theorem 5 can be completed as follows: pipelining the presented reduction with an algorithm for C_4-Free Completion working in $2^{o(k)} n^{\mathcal{O}(1)}$ time would give an algorithm for 3Sat working in $2^{o(n+m)}(n+m)^{\mathcal{O}(1)}$ time, which contradicts ETH

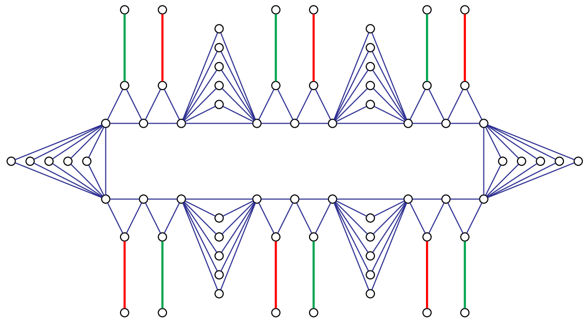


Figure 13: Variable gadget G_x for a variable appearing in six clauses in φ, i.e., $p_x = 6$. Deleting the leftmost edges in each tower pair corresponds to setting x to false and deleting the rightmost edge in each tower pair corresponds to setting x to true.

by the results of Impagliazzo, Paturi and Zane [14].

6.3 P_4-free completion is not solvable in subexponential time

In this section we show that there is no subexponential algorithm for $\mathcal{F}$-Completion for $\mathcal{F} = \{P_4\}$ unless the ETH fails. Let us recall that since $\overline{P_4} = P_4$, the problems P_4-Free Edge Deletion and P_4-Free Completion are polynomial time equivalent. In other words, we aim to convince the reader of the following.

Theorem 6. *The problem P_4-Free Completion is not solvable in $2^{o(k)} n^{\mathcal{O}(1)}$ time unless ETH fails.*

We reduce from 3Sat to the complement problem P_4-Free Edge Deletion. Let φ be the input 3-CNF formula, where we again assume that every clause of φ contains exactly three literals corresponding to pairwise different variables. For a variable $x \in \mathcal{V}(\varphi)$ we denote by p_x the number of clauses in φ containing x. Note that since each clause contains exactly three variables, we have that $\sum_{x \in \mathcal{V}(\varphi)} p_x = 3|\mathcal{C}(\varphi)|$. We construct a graph G_φ such that for $k_\varphi = 4|\mathcal{C}(\varphi)| + \sum_{x \in \mathcal{V}(\varphi)} 4p_x = 16|\mathcal{C}(\varphi)|$, φ is satisfiable if and only if (G_φ, k_φ) is a ***yes*** instance of P_4-Free Edge Deletion. Since the complement of P_4 is again P_4, this will prove the theorem.

Variable gadget. For each variable $x \in \mathcal{V}(\varphi)$, we create a gadget G^x which looks like the one given in Figure 13. Before providing the construction formally, let us first describe it informally. We call a triangle with a pendant vertex a *tower*, where the triangle will be referred to as the *base* of the tower, and the pendant vertex the *spike* of the tower. The towers will always come in pairs, and they are joined in one of the vertices in the bases (two vertices are identified, again see the figure). Pairs of towers will be separated by k triangles sharing an edge. The vertices not shared between the k triangles will be called the *stack*, whereas the edge shared among the triangles will be called the *shortcut*.

The gadget G^x for a variable x consists of p_x pairs of towers arranged on a cycle, one for each clause x appears in, where every two consecutive pairs on the cycle are separated by a shortcut edge and a stack of vertices. The stack is chosen to be big enough ($k' = k_\varphi + 3$ vertices) so that we will never delete the edge that connects the two towers on each side of the stack, nor any edge incident to a vertex from the stack. We will refer to the two towers in the pairs as Tower 1 (the one with lower index) and Tower 2.

Formally, let φ be an instance of 3Sat. The budget for the output instance will be $k_\varphi = 4|\mathcal{C}(\varphi)| + \sum_{x \in \mathcal{V}(\varphi)} 4p_x = 16|\mathcal{C}(\varphi)|$. Let $k' = k_\varphi + 3$. For a variable x which appears in p_x clauses, we create vertices $s^x_{i,j}$ for $i \in \{1,\ldots,p_x\}$ and $j \in \{1,\ldots,k'\}$. These will be the vertices for the stacks. For the spikes of the towers, we add vertices $t^x_{i,1}$ and $t^x_{i,2}$ for $i \in \{1,\ldots,p_x\}$. For the base of the towers, we add vertices $b^x_{i,j}$ for $j \in \{1,\ldots,5\}$ and $i \in \{1,\ldots,p_x\}$. These are all the vertices of the gadget G^x for $x \in \mathcal{V}(\varphi)$.

The vertices denoted by t are the two spikes in the tower, i.e., $t^x_{i,1}$ is the spike of the Tower 1 of the ith pair for variable x. The vertices denoted by b are for the bases (there are five vertices in the bases of the two towers).

Now we add the edges to G^x, see Figure 14:

- For the stack, we add edges $s^x_{i,j}b^x_{i,1}$ for all $i \in \{1,\ldots,p_x\}$ and $j \in \{1,\ldots,k'\}$ (right side of the stack) and edges $b^x_{i,5}s^x_{i+1,j}$ for $i \in \{1,\ldots,p_x\}$ and $j \in \{1,\ldots,k'\}$ (the left side of the next stack), where the indices behave cyclically modulo p_x.
- For the bases, we add the edges $b^x_{i,1}b^x_{i,2}$, $b^x_{i,1}b^x_{i,3}$, $b^x_{i,2}b^x_{i,3}$, $b^x_{i,3}b^x_{i,4}$, $b^x_{i,3}b^x_{i,5}$ and $b^x_{i,4}b^x_{i,5}$ for $i \in \{1,\ldots,p_x\}$. To attach the towers, we add the edges $b^x_{i,2}t^x_{i,1}$ and $b^x_{i,4}t^x_{i,2}$. The set of these eight edges will be denoted by R^x_i.
- The last edges to add are the shortcut edges $b^x_{i,5}b^x_{i+1,1}$ for $i \in \{1,\ldots,p_x\}$, where again the indices behave cyclically modulo p_x.

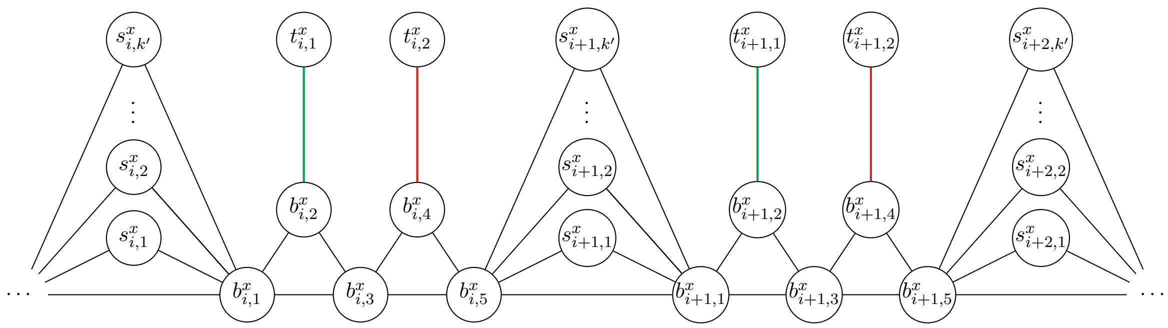


Figure 14: Variable gadget G^x. The counter i ranges from 1 to p_x, the number of clauses x appears in. This figure does not illustrate that the gadget is a cycle, see Figure 13 for a zoomed-out version.

Elimination from variable gadgets. We will now show that there are exactly two ways of eliminating all P_4s occurring in a variable gadget using at most $4p_x$ edges. To state this claim formally, we need to control how the variable gadget is situated in a larger construction of the whole output instance that will be defined later. We say that a variable gadget G^x is *properly embedded* in the output instance G_φ if G^x is an induced subgraph of G_φ, and moreover the only vertices of G^x that are incident to edges outside G^x are the spikes of the towers, i.e., vertices $t^x_{i,1}$ and $t^x_{i,2}$ for $i \in \{1,2,\ldots,p_x\}$. This property will be satisfied for gadgets G_x for all $x \in \mathcal{V}(\varphi)$ in the next steps of the construction. Using this notion, we can infer properties of the variable gadget irrespective of the whole output instance G_φ constructed later looks like.

We first show that an inclusion minimal deletion set S that has size at most k_φ cannot touch the stacks nor the shortcut edges.

Claim 6.6. *Assume gadget G^x is embedded properly in the output graph G_φ, and that S is an inclusion minimal P_4-free edge deletion set in G_φ of size at most k_φ. Then S does not contain any edge of type $b^x_{i,5}b^x_{i+1,1}$ (a shortcut edge), nor any edge incident to a vertex of the form $s^x_{i,j}$.*

Proof of claim. (See Figure 14 for indices.)

Suppose first that a shortcut edge $b^x_{i,5}b^x_{i+1,1}$ belongs to S. Let $S' = S \setminus \{b^x_{i,5}b^x_{i+1,1}\}$. Since S was inclusion minimal, the graph $G_\varphi - S'$ must contain an induced P_4 that contains the edge $b^x_{i,5}b^x_{i+1,1}$; denote this P_4 by L. By the assumption that G^x is properly embedded in G_φ we have that L is entirely contained in G^x. Since the stack between pairs of towers i and $i+1$ has height $k' = k_\varphi + 3$, we know that there are at least three vertices of the form $s^x_{i+1,j}$ for some $j \le k$ which are not incident to an edge in S. Since L passes through 2 vertices apart from $b^x_{i,5}$ and $b^x_{i+1,1}$, we infer that one of these vertices, say s^x_{i+1,j_0}, is not incident to any edge of S, nor it lies on L. Create L' by replacing the edge $b^x_{i,5}b^x_{i+1,1}$ with the path $b^x_{i,5} - s^x_{i+1,j_0} - b^x_{i+1,1}$ on L. We infer that L' is an induced P_5 in $G_\varphi - S$, which in particular contains an induced P_4. This is a contradiction to the definition of S.

Second, without loss of generality suppose now that the edge $b^x_{i,5}s^x_{i+1,j}$ belongs to S for some $j \in \{1, 2, \ldots, k'\}$. Let $S' = S \setminus \{b^x_{i,5}s^x_{i+1,j}, b^x_{i+1,1}s^x_{i+1,j}\}$; note here that the edge $b^x_{i+1,1}s^x_{i+1,j}$ might had not belonged to S, but if it had, then we remove it when constructing S'. Since S was inclusion minimal, the graph $G_\varphi - S'$ must contain an induced P_4 that contain the vertex $s^x_{i+1,j}$, so also one of the vertices $b^x_{i,5}$ or $b^x_{i+1,1}$; denote this P_4 by L. Again, by the definition of proper embedding we have that L is entirely contained in G^x. By the same argumentation as before we infer that there exists a vertex s^x_{i+1,j_0} such that s^x_{i+1,j_0} is not traversed by L and is not incident to an edge of S. Since vertices s^x_{i+1,j_0} and $s^x_{i+1,j}$ are twins in $G_\varphi - S'$, it follows that the path L' constructed from L by substituting $s^x_{i+1,j}$ with s^x_{i+1,j_0} is an induced P_4 in $G_\varphi - S$. This is a contradiction to the definition of S. □

Now we show that every minimal deletion set S must use at least 4 edges in each pair of towers, and if it uses exactly 4 edges then there are exactly 4 ways how the intersection of S with this pair of towers can look like.

Claim 6.7. *Assume that the gadget G^x is embedded properly in the output graph G_φ, and that S is an inclusion minimal P_4-free edge deletion set in G_φ of size at most k_φ. Then for each $i \in \{1, 2, \ldots, p_x\}$ it holds that $|R^x_i \cap S| \ge 4$, and if $|R^x_i \cap S| = 4$ then either:*

Elimination A: $R^x_i \cap S$ consists of the edges of the base of Tower 1 and the spike of Tower 2, or

Elimination B: $R^x_i \cap S$ consists of the edges of the base of Tower 2 and the spike of Tower 1, or

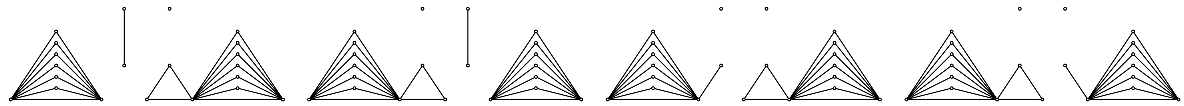
Elimination C: $R^x_i \cap S$ consists of the edges of both spikes and of the base of Tower 1 apart from the edge $b^x_{i,1}b^x_{i,2}$, or

Elimination D: $R^x_i \cap S$ consists of the edges of both spikes and of the base of Tower 2 apart from the edge $b^x_{i,4}b^x_{i,5}$.

We refer to Figure 15 for visualization of all the four types of eliminations. We will say that $R^x_i \cap S$ *realizes Elimination* X for X being A, B, C, or D, if $R^x_i \cap S$ is as described in the statement of Claim 6.7. Similarly, we say that the ith pair of towers *realizes Elimination* X if $R^x_i \cap S$ does.

Proof of Claim 6.7. By Claim 6.6 we infer that S does not contain any edge incident to stacks i and $i+1$, nor any of the shortcut edges incident to the considered pair of towers. We consider four cases, depending on how the set $S \cap \{t^x_{i,1}b^x_{i,2}, t^x_{i,2}b^x_{i,4}\}$ looks like. In each case we prove that $|R^x_i \cap S| \ge 4$, and that $|R^x_i \cap S| = 4$ implies that one of four listed elimination types is used.

First assume that $S \cap \{t^x_{i,1}b^x_{i,2}, t^x_{i,2}b^x_{i,4}\} = \emptyset$ and observe that $s^x_{i,1} - b^x_{i,1} - b^x_{i,2} - t^x_{i,1}$ and $s^x_{i+1,1} - b^x_{i,5} - b^x_{i,4} - t^x_{i,2}$ are induced P_4s in G_φ. Since on each of these P_4s there is only one



(a) Elimination A (false) (b) Elimination B (true) (c) Elimination C (d) Elimination D

Figure 15: The four different ways of eliminating a tower pair in a variable gadget. Only Eliminations A and B yield optimum deletion sets in an entire variable gadget. They all use exactly four edges per pair of towers, as is evident in the figure.

edge that is not assumed to be not belonging to S, it follows that both $b^x_{i,1}b^x_{i,2}$ and $b^x_{i,4}b^x_{i,5}$ must belong to S. Suppose that $b^x_{i,1}b^x_{i,3} \notin S$. Then we infer that $b^x_{i,2}b^x_{i,3}, b^x_{i,3}b^x_{i,4}, b^x_{i,3}b^x_{i,5} \in S$, since otherwise any of these edges would form an induced P_4 in $G_\varphi - S$ together with edges $b^x_{i,1}b^x_{i,3}$ and $b^x_{i,1}s^x_{i,1}$. We infer that in this case $|R^x_i \cap S| \geq 5$, and a symmetric conclusion can be drawn when $b^x_{i,3}b^x_{i,5} \notin S$. We are left with the case when $b^x_{i,1}b^x_{i,3}, b^x_{i,3}b^x_{i,5} \in S$. But then S must include also one of the edges $b^x_{i,2}b^x_{i,3}$ or $b^x_{i,3}b^x_{i,4}$ so that the induced P_4 $t^x_{i,1} - b^x_{i,2} - b^x_{i,3} - b^x_{i,4}$ is destroyed. Hence, in all the considered cases we conclude that $|R^x_i \cap S| \geq 5$.

Second, assume that $S \cap \{t^x_{i,1}b^x_{i,2}, t^x_{i,2}b^x_{i,4}\} = \{t^x_{i,2}b^x_{i,4}\}$. The same reasoning as in the previous paragraph shows that $b^x_{i,1}b^x_{i,2}$ must belong to S. Again, if $b^x_{i,1}b^x_{i,3} \notin S$, then all the edges $b^x_{i,2}b^x_{i,3}, b^x_{i,3}b^x_{i,4}, b^x_{i,3}b^x_{i,5}$ must belong to S, and so $|R^x_i \cap S| \geq 5$. Assume then that $b^x_{i,1}b^x_{i,3} \in S$. Note now that we have two induced P_4*s*: $t^x_{i,1} - b^x_{i,2} - b^x_{i,3} - b^x_{i,4}$ and $t^x_{i,1} - b^x_{i,2} - b^x_{i,3} - b^x_{i,5}$ that share the edge $t^x_{i,1}b^x_{i,2}$ about which we assumed that it does not belong to S, and the edge $b^x_{i,2}b^x_{i,3}$. To remove both these P_4s we either remove at least two more edges, which results in conclusion that $|R^x_i \cap S| \geq 5$, or remove the edge $b^x_{i,2}b^x_{i,3}$, which results in Elimination A.

The third case when $S \cap \{t^x_{i,1}b^x_{i,2}, t^x_{i,2}b^x_{i,4}\} = \{t^x_{i,1}b^x_{i,2}\}$ is symmetric to the second case, and leads to a conclusion that either $|R^x_i \cap S| \geq 5$ or $R^x_i \cap S$ realizes Elimination B.

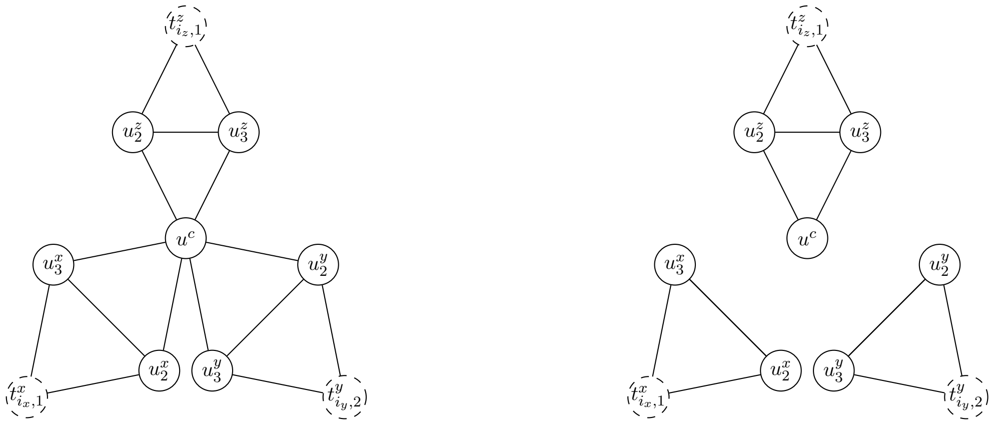
Finally, assume that $t^x_{i,1}b^x_{i,2}, t^x_{i,2}b^x_{i,4} \in S$. Observe that we have an induced P_4 $s^x_{i,1} - b^x_{i,1} - b^x_{i,3} - b^x_{i,5}$ in G_φ, so one of the edges $b^x_{i,1}b^x_{i,3}$ or $b^x_{i,3}b^x_{i,5}$ must be included in S. Assume first that $b^x_{i,1}b^x_{i,3} \in S$. Consider now P_4s $s^x_{i,1} - b^x_{i,1} - b^x_{i,2} - b^x_{i,3}$ and $b^x_{i,2} - b^x_{i,3} - b^x_{i,5} - s^x_{i+1,1}$. Both these P_4s need to be destroyed by S since after removing $b^x_{i,1}b^x_{i,3}$ the first P_4 becomes induced, while the second is induced already in G_φ. Moreover, these P_4s share only the edge $b^x_{i,2}b^x_{i,3}$, which means that either $|R^x_i \cap S| \geq 5$ or $b^x_{i,2}b^x_{i,3} \in S$ and $R^x_i \cap S$ realizes Elimination C. The case when $b^x_{i,3}b^x_{i,5} \in S$ is symmetric and leads to a conclusion that either $|R^x_i \cap S| \geq 5$ or $R^x_i \cap S$ realizes Elimination D. □

Finally, we are able to prove that the variable gadget G^x requires at least $4p_x$ edge deletions, and that there are only two ways of destroying all P_4s by using exactly $4p_x$ edge deletions: either by applying Elimination A or Elimination B to all the pairs of towers.

Claim 6.8. *Suppose a gadget G^x is embedded properly in the output graph G_φ, and that S is an inclusion minimal P_4-free edge deletion set in G_φ of size at most k_φ. Then $|E(G^x) \cap S| \geq 4p_x$, and if $|E(G^x) \cap S| = 4p_x$, then either $R^x_i \cap S$ realizes Elimination A for all $i \in \{1, 2, \ldots, p_x\}$, or $R^x_i \cap S$ realizes Elimination B for all $i \in \{1, 2, \ldots, p_x\}$.*

Proof of claim. By Claims 6.6 and 6.7 we have that S does not contain any shortcut edge or edge incident to a stack vertex, and moreover that $|R^x_i \cap S| \geq 4$ for all $i \in \{1, 2, \ldots, p_x\}$. Since sets R^x_i are pairwise disjoint, it follows that $|E(G^x) \cap S| \geq 4p_x$. Moreover, if $|E(G^x) \cap S| = 4p_x$, then $|R^x_i \cap S| = 4$ for all $i \in \{1, 2, \ldots, p_x\}$ and, by Claim 6.7, for all $i \in \{1, 2, \ldots, p_x\}$ the set $R^x_i \cap S$ must realize Elimination A, B, C, or D.

We say that one pair of towers is *followed* by another, if the former has index i, and the latter has index $i + 1$ (of course, modulo p_x). To obtain the conclusion that either all



(a) Clause gadget for a clause $c = x \lor \neg y \lor z$. The dashed vertices are the connection points in the variable gadgets. Observe that since y appears negated, we attach it to Tower 2 in its pair. The clause c is the i_ℓth clause ℓ appears in.

(b) Clause gadget elimination when c is satisfied by variable z.

Figure 16: Clause gadget G^c for a clause $c = x \lor \neg y \lor z$. To the left it is before elimination, to the right an optimal elimination when satisfied by z.

the sets $R_i^x \cap S$ realize Elimination A or all of them realize Elimination B, we observe that when some pair of towers realize Elimination A, C, or D, then the following pair must realize Elimination A. Indeed, otherwise the graph $G_\varphi - S$ would contain an induced P_4 of the form $b_{i,4}^x - b_{i,5}^x - b_{i+1,1}^x - b_{i+1,3}^x$, where the ith pair of towers is the considered pair that realizes Elimination A, C, or D. Now observe that since the pairs of towers are arranged on a cycle, then either all pairs of towers realize Elimination B, or at least one realizes Elimination A, C, or D, which means that the following pair realizes Elimination A, and so all the pairs must realize Elimination A. ☐

Clause gadget. We now move on to construct the clause gadget G^c for a clause $c \in \mathcal{C}(\varphi)$. Assume that $c = \ell_x \lor \ell_y \lor \ell_z$, where ℓ_r is a literal of variable r for $r \in \{x, y, z\}$. We create seven vertices: one vertex u^c and vertices u_2^r and u_3^r for $r = x, y, z$. We also add the edges $u^c u_2^r$, $u^c u_3^r$ and $u_2^r u_3^r$. Now, for non-negated $r \in \{x, y, z\}$ in c, where c is the ith clause r appears in, we add edges $u_2^r t_{i,1}^r$ and $u_3^r t_{i,1}^r$ (recall that $t_{i,1}^r$ is the spike of Tower 1 in tower pair i). If r appears negated, we add the edges $u_2^r t_{i,2}^r$ and $u_3^r t_{i,2}^r$ instead, see Figure 17. Let M^c be the set comprising all the 15 created edges, including the ones incident to the spikes of the towers. By $M^{c,r}$ for $r \in \{x, y, z\}$ we denote the subset of M^c containing 5 edges that are incident to vertex u_2^r or u_3^r.

This concludes the construction of the graph G_φ; note that all the variable gadgets are properly embedded in G_φ. Before showing the correctness of the reduction, we prove the following claims about the number of edges needed for the clause gadgets:

Claim 6.9. *Assume that S is a P_4-free deletion set of graph G_φ. Let c be a clause of φ, and assume that x, y, z are the variables appearing in c. Then $|S \cap M^c| \geq 4$, and if $|S \cap M^c| = 4$, then $S \cap M^{c,r} = \emptyset$ for some $r \in \{x, y, z\}$ (see Figure 16b for an example where $S \cap M^{c,z} = \emptyset$).*

Proof of claim. To simplify the notation, let t^x, t^y, t^z be the corresponding vertices of the variable gadgets that are incident to edges of M^c.

If $|S \cap M^{c,r}| \geq 2$ for all $r \in \{x, y, z\}$, then $|S \cap M^c| \geq 6$ and we are done. Assume then without loss of generality that $|S \cap M^{c,x}| \leq 1$. Hence, at least one of the paths $t^x - u_2^x - u^c$

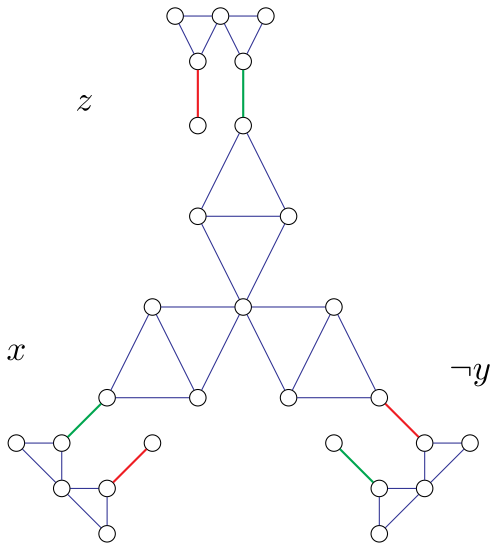


Figure 17: For a clause $c = x \vee \neg y \vee z$, we obtain the above connection. For negated variables, the rightmost spike is attached to the gadgets, otherwise the leftmost spike is attached. If x is being evaluated to a value satisfying c, the edge spike between G^x and G^c is deleted.

and $t^x - u_3^x - u^c$ does not contain an edge of S. Assume without loss of generality that it is $t^x - u_2^x - u^c$. Now observe that in G_φ we have 4 induced P_4 created by prolonging this P_3 by vertex u_2^y, u_3^y, u_2^z or u_3^z. Since $t^x - u_2^x - u^c$ is disjoint with S, it follows that all the four edges connecting these vertices with u^c must belong to S. Hence $|S \cap M^c| \geq 4$, and if $|S \cap M^c| = 4$ then $M^{c,x}$ must be actually disjoint with S. $\square$

We are finally ready to prove the following lemma, which implies correctness of the reduction.

Lemma 6.10. *Given an input instance φ to* 3Sat*, φ is satisfiable if and only if the constructed graph G_φ has a P_4 deletion set of size $k_\varphi = 16|\mathcal{C}(\varphi)|$.*

Proof. From left to right, suppose φ is satisfiable by an assignment α, and let G_φ and k_φ be as above. If a variable x is assigned false in α, we delete as in Figure 15a, that is, we apply Elimination A to all the pairs of towers in the variable gadget G^x. Otherwise we delete as in Figure 15b, that is, we apply Elimination B to all the pairs of towers in the variable gadget G^x. In other words, if x assigned to false (Elimination A), we delete the edges $t_{i,2}^x b_{i,4}^x$, $b_{i,1}^x b_{i,2}^x$, $b_{i,1}^x b_{i,3}^x$, $b_{i,2}^x b_{i,3}^x$, otherwise, when x is assigned to true (Elimination B), we delete the edges $t_{i,1}^x b_{i,2}^x$, $b_{i,3}^x b_{i,4}^x$, $b_{i,3}^x b_{i,5}^x$, $b_{i,4}^x b_{i,5}^x$, for all $i \in \{1, \ldots, p_x\}$.

Furthermore, for every clause $c = \ell_x \vee \ell_y \vee \ell_z$ we choose an arbitrary variable whose literal satisfies c, say r. We remove the edges $u_2^{r'} u^c$ and $u_3^{r'} u^c$ for $r' \neq r$. We have thus used exactly four edge removals per clause, $4|\mathcal{C}(\varphi)|$ in total, and for each $x \in \mathcal{V}(\varphi)$ we have removed $4p_x$ edges. This sums up exactly to $4|\mathcal{C}(\varphi)| + \sum_{x \in \mathcal{V}(\varphi)} 4p_x = 4|\mathcal{C}(\varphi)| + 4\sum_{x \in \mathcal{V}(\varphi)} p_x = 4|\mathcal{C}(\varphi)| + 4 \cdot 3|\mathcal{C}(\varphi)| = 16|\mathcal{C}(\varphi)| = k_\varphi$ edge removals.

We now claim that G_φ is P_4-free. A direct check shows that there is no induced P_4 left inside any variable gadget, nor inside any clause gadget. Therefore, any induced P_4 left must necessarily contain vertex of the form $t_{i,q}^x$ for some $x \in \mathcal{V}(\varphi)$, $i \in \{1, 2, \ldots, p_x\}$, and $q \in \{1, 2\}$, together with the edge of the spike incident to this vertex and one of the edges of gadget G^c incident to this vertex, where c is the ith clause x appears in. Assume without loss of generality that $q = 1$, so x appears in c positively. Since we did not delete the spike edge $t_{i,1}^x b_{i,2}^x$, we infer that $\alpha(x) = \texttt{false}$. Therefore x does not satisfy c, so we must have deleted edges $u^c u_2^x$ and

$u^c u_3^x$. Thus in the remaining graph $G_\varphi - S$ the connected component of the vertex $t_{i,q}^x$ is a triangle with a pendant edge, which is P_4-free. We conclude that $G_\varphi - S$ is indeed P_4-free.

From right to left, suppose now that G_φ is the constructed graph from a fixed φ and that for k_φ as above, we have that (G_φ, k_φ) is a ***yes*** instance of P_4-Free Edge Deletion. Let S be a P_4 deletion set of size at most k_φ, and without loss of generality assume that S is inclusion minimal. By Claims 6.8 and 6.9 we infer that S must contain at least $4p_x$ edges in each set $E(G^x)$, and at least four edges in each set M^c. Since $4|\mathcal{C}(\varphi)| + \sum_{x \in \mathcal{V}(\varphi)} 4p_x = k_\varphi$, we infer that S contains exactly $4p_x$ edges in each set $E(G^x)$, and exactly four edges in each set M^c. By Claim 6.8 we infer that for each variable x, all the pairs of towers in G^x realize Elimination A, or all of them realize Elimination B. Let $\alpha\colon \mathcal{V}(\varphi) \to \{\texttt{true}, \texttt{false}\}$ be an assignment that assigns value `false` if Elimination A is used throughout the corresponding gadget, and value `true` otherwise. We claim that α satisfies φ.

Consider a clause $c \in \mathcal{C}(\varphi)$ and assume that x, y, z are variables appearing in c. By Claim 6.9 we infer that there exists $r \in \{x, y, z\}$ such that $S \cap M^{c,r} = \emptyset$. Assume without loss of generality that $r = x$, and that x appears positively in c. Moreover, assume that c is the i_xth clause x appears in. We claim that $\alpha(x) = \texttt{true}$, and thus c is satisfied by x. Indeed, otherwise the edge $t_{i_x,1}^x b_{i_x,2}^x$ would not be deleted, and thus $b_{i_x,2}^x - t_{i_x,1}^x - u_2^x - u^c$ would be an induced P_4 in $G_\varphi - S$; this is a contradiction to the definition of S. ☐

Again, the proof of Theorem 6 follows: pipelining the presented reduction with an algorithm for P_4-Free Edge Deletion working in $2^{o(k)} n^{\mathcal{O}(1)}$ time would give an algorithm for 3Sat working in $2^{o(n+m)}(n+m)^{\mathcal{O}(1)}$ time, which contradicts ETH by the results of Impagliazzo, Paturi and Zane [14].

It is easy to verify that in the presented reduction, both the graph G_φ and $G_\varphi - S$ for S being the deletion set constructed for a satisfying assignment for φ are actually C_4-free. Thus the same reduction also shows that $\{C_4, P_4\}$-free Deletion is not solvable in $2^{o(k)} n^{\mathcal{O}(1)}$ time unless ETH fails. For $\mathcal{F} = \{2K_2, P_4\}$, we refer to $\mathcal{F}$-Completion as to $\{2K_2, P_4\}$-Free Completion. Since the complement of $\{2K_2, P_4\}$ is $\{C_4, P_4\}$, we derive the following result.

Theorem 7. *The problem* $\{2K_2, P_4\}$-Free Completion *is not solvable in* $2^{o(k)} n^{\mathcal{O}(1)}$ *time unless ETH fails.*

In other words, Co-Trivially Perfect Completion, or, equivalently, Trivially Perfect Edge Deletion, is not solvable in subexponential time unless ETH fails.

7 Conclusion and open problems

In this paper, we provided several upper and lower subexponential parameterized bounds for $\mathcal{F}$-Completion. The most natural open question would be to ask for a dichotomy type of result characterizing for which sets $\mathcal{F}$, $\mathcal{F}$-Completion problems are in P, in SUBEPT, and not in SUBEPT (under ETH). Keeping in mind the lack of such characterization concerning classes P and NP, an answer to this question can be very non-trivial. Even a more modest task—deriving general arguments explaining what makes a completion problem to be in SUBEPT—is an important open question.

Similarly, from an algorithmic perspective, obtaining generic subexponential algorithms for completion problems would be a big step forwards. With the current knowledge, for different cases of $\mathcal{F}$, the algorithms are built on different ideas like chromatic coding, potential maximal cliques, k-cuts, etc. and each new case requires special treatment.

Finally, some concrete problems. We have the chain of graph classes

$$\textbf{threshold} \subset \textbf{trivially perfect} \subset \textbf{interval} \subset \textbf{chordal},$$

corresponding to the parameters vertex cover, treedepth, pathwidth, and treewidth, in the sense that the width parameter is the minimum, over all completions to the graph class mentioned, size of the maximum clique (± 1). We know that all of these problems have subexponential completion problems, except for Interval Completion. The problem is known to be FPT [23]. It is natural to ask whether or not this problem also belongs to SUBEPT.

Another chain of graph classes and width parameters is

$$\textbf{proper interval} \subset \textbf{interval} \subset \textbf{chordal},$$

corresponding to bandwidth, pathwidth and treewidth. The existence of a subexponential algorithm for Proper Interval Completion is also open.

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