# Detecting Induced Minors in AT-free Graphs *

Petr A. Golovach[1], Dieter Kratsch[2], and Daniël Paulusma[3]

[1] Department of Informatics, Bergen University, PB 7803, 5020 Bergen, Norway, petr.golovach@ii.uib.no

[2] Laboratoire d'Informatique Théorique et Appliquée, Université de Lorraine, 57045 Metz Cedex 01, France, kratsch@univ-metz.fr

[3] School of Engineering and Computing Sciences, Durham University, South Road, Durham DH1 3LE, UK, daniel.paulusma@durham.ac.uk

**Abstract.** The problem Induced Minor is to test whether a graph $G$ can be modified into a graph $H$ by a sequence of vertex deletions and edge contractions. We prove that Induced Minor is polynomial-time solvable when $G$ is AT-free, and $H$ is fixed, i.e., not part of the input. Our result can be considered to be optimal in some sense as we also prove that Induced Minor is W[1]-hard on AT-free graphs, when parameterized by $|V_H|$. In order to obtain it we prove that the Set-Restricted $k$-Disjoint Paths problem can be solved in polynomial time on AT-free graphs for any fixed $k$. We also use the latter result to prove that the Set-Restricted $k$-Disjoint Connected Subgraphs problem is polynomial-time solvable on AT-free graphs for any fixed $k$.

## 1 Introduction

In this paper we study graph containment problems. Whether or not a graph contains some other graph depends on the notion of containment used. In the literature several natural definitions have been studied such as containing a graph as a contraction, dissolution, immersion, (induced) minor, (induced) topological minor, (induced) subgraph, or (induced) spanning subgraph. We focus on the containment relation "induced minor". A graph $G$ contains a graph $H$ as an *induced minor* if $G$ can be modified into a graph $H$ by a sequence of vertex deletions and edge contractions. Here, the operation *edge contraction* removes the end-vertices $u$ and $v$ of an edge from $G$ and replaces them by a new vertex adjacent to precisely those vertices to which $u$ or $v$ were adjacent. The corresponding decision problem asking whether $H$ is an induced minor of $G$ is called Induced Minor. This problem is NP-complete even when $G$ and $H$ are trees of bounded diameter or trees, the vertices of which have degree at most 3 except for at most one vertex, as shown by Matoušek and Thomas [14]. It is therefore natural to fix the graph $H$ and to consider only the graph $G$ to be part of the input. We denote this variant as $H$-Induced Minor.

The computational complexity classification of $H$-Induced Minor is far from being settled, although both polynomial-time and NP-complete cases are

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known. In contrast, the two related problems $H$-Minor and $H$-Topological Minor, which are to test whether a graph $G$ contains a graph $H$ as a minor or topological minor, respectively, can be solved in cubic time for any fixed graph $H$, as shown by Robertson and Seymour [16] and Grohe et al. [9], respectively. Fellows et al. [5] showed that there exists a graph $H$ for which $H$-Induced Minor is NP-complete. This specific graph $H$ has 68 vertices and is still the smallest $H$ for which $H$-Induced Minor is known to be NP-complete. The question whether $H$-Induced Minor is polynomial-time solvable for any fixed tree $H$ was posed as an open problem at the AMS-IMS-SIAM Joint Summer Research Conference on Graph Minors in 1991. So far this question could only be answered for trees on at most 7 vertices except for one case [6].

Due to the notorious difficulty of solving $H$-Induced Minor for general graphs, the input has been restricted to special graph classes. Fellows et al. [5] showed that for every fixed graph $H$, the $H$-Induced Minor problem can be solved in linear time on planar graphs. Van 't Hof et al. [10] extended this result by proving that for every fixed planar graph $H$, the $H$-Induced Minor problem is polynomial-time solvable on any minor-closed graph class not containing all graphs. Belmonte et al. [1] showed that for every fixed graph $H$, the $H$-Induced Minor problem is polynomial-time solvable for chordal graphs, whereas for claw-free graphs a number of partial results, which only include polynomial-time solvable cases, are known [7].

We consider $H$-Induced Minor restricted to the class of *asteroidal triple-free* graphs, also known as *AT-free* graphs. An *asteroidal triple* is a set of three mutually non-adjacent vertices such that each two of them are joined by a path that avoids the neighborhood of the third, and AT-free graphs are exactly those graphs that contain no such triple. AT-free graphs, defined fifty years ago by Lekkerkerker and Boland [13], are well studied in the literature and contain many well-known classes, e.g., cobipartite graphs, cocomparability graphs, cographs, interval graphs, permutation graphs, and trapezoid graphs (cf. [3]). All these graph classes have geometric intersection models being extremely useful when designing polynomial-time algorithms for hard problems. No such model is available for AT-free graphs. Recently, Golovach et al. [8] showed that the $H$-Induced Topological Minor problem is polynomial-time solvable on AT-free graphs for every fixed $H$. They also showed that this problem is W[1]-hard when parameterized by $|V_H|$.

**Our Results.** We show that $H$-Induced Minor can be solved in polynomial time on AT-free graphs for any fixed graph $H$. Consequently, on AT-free graphs, all four problems $H$-Minor, $H$-Induced Minor, $H$-Topological Minor and $H$-Induced Topological Minor are polynomial-time solvable for any fixed graph $H$. In addition, we prove that Induced Minor is W[1]-hard when parameterized by $|V_H|$. Our proof also implies the NP-completeness of Induced Minor for AT-free graphs, which was not known before.

The celebrated result by Robertson and Seymour that $H$-Minor is FPT on general graphs [16] is closely connected to the fact that $k$-Disjoint Paths is FPT with parameter $k$. To solve $H$-Induced Topological Minor on AT-free graphs, Golovach et al. [8] considered the variant $k$-Induced Disjoint Paths, in which the paths must not only be vertex-disjoint but also mutually induced, i.e., edges between vertices of any two distinct paths are forbidden. Here we must consider another variant, which was introduced by Belmonte et al. [1]. A *terminal pair* in a graph $G = (V, E)$ is a specified pair of vertices $s$ and $t$ called *terminals*, and the *domain* of a terminal pair $(s, t)$ is a specified subset $U \subseteq V$ containing both $s$ and $t$. We say that two paths, each of which is between some terminal pair, are *vertex-disjoint* if they have no common vertices except possibly the vertices of the terminal pairs. This leads to the following decision problem, which is NP-complete on general graphs even when $k = 2$ [1].

Set-Restricted $k$-Disjoint Paths
*Instance:* a graph $G$, terminal pairs $(s_1, t_1), \ldots, (s_k, t_k)$, and domains $U_1, \ldots, U_k$.
*Question:* does $G$ contain $k$ mutually vertex-disjoint paths $P_1, \ldots, P_k$ such that $P_i$ is a path from $s_i$ to $t_i$ using only vertices from $U_i$ for $i = 1, \ldots, k$?

Note that the domains $U_1, \ldots, U_k$ are not necessarily pairwise disjoint. If we let every domain contain all vertices of $G$, we obtain exactly the Disjoint Paths problem. We give an algorithm that solves Set-Restricted $k$-Disjoint Paths in polynomial time on AT-free graphs for any fixed integer $k$. We then use this algorithm as a subroutine in our polynomial-time algorithm that solves $H$-Induced Minor on AT-free graphs for any fixed graph $H$. We emphasize that we can not apply the algorithm for $k$-Induced Disjoint Paths on AT-free graphs [8] as a subroutine to solve $H$-Induced Minor on AT-free graphs. Also, the techniques used in that algorithm are quite different from the techniques we use here to solve Set-Restricted $k$-Disjoint Paths on AT-free graphs. Moreover, when $k$ is in the input, $k$-Induced Disjoint Paths and Set-Restricted $k$-Disjoint Paths have a different complexity for AT-free graphs. Golovach et al. [8] proved that in that case $k$-Induced Disjoint Paths is polynomial-time solvable for AT-free graphs, whereas $k$-Disjoint Paths, and consequently Set-Restricted $k$-Disjoint Paths, is already NP-complete for interval graphs [15], a subclass of AT-free graphs.

We use our algorithm for solving Set-Restricted $k$-Disjoint Paths to obtain two additional results on AT-free graphs. The first result is that we can solve the problem Set-Restricted $k$-Disjoint Connected Subgraphs in polynomial time on AT-free graphs for any fixed integer $k$. A *terminal set* in a graph $G = (V, E)$ is a specified subset $S_i \subseteq V$.

Set-Restricted $k$-Disjoint Connected Subgraphs
*Instance:* a graph $G$, terminal sets $S_1, \ldots, S_k$, and domains $U_1, \ldots, U_k$.
*Question:* does $G$ have $k$ pairwise vertex-disjoint connected subgraphs $G_1, \ldots, G_k$, such that $S_i \subseteq V_{G_i} \subseteq U_i$, for $1 \leq i \leq k$?

If $|S_i| = 2$ for all $1 \leq i \leq k$, then we obtain the Set-Restricted $k$-Disjoint Paths problem. If $U_i = V_G$ then we obtain the $k$-Disjoint Connected Subgraphs problem. The latter problem has been introduced by Robertson and Seymour [16] and is NP-complete on general graphs even when $k = 2$ and $\min\{|Z_1|, |Z_2|\} = 2$ [11]. The second result is that we can solve the problem $H$-Contractibility in polynomial time on AT-free graphs for any fixed triangle-free graph $H$. This problem is to test whether a graph $G$ can be modified into a graph $H$ by a sequence of contractions only. For general graphs, its complexity classification is still open but among other things it is known that the problem is already NP-complete when $H$ is the 4-vertex path or the 4-vertex cycle [2].

## 2 Preliminaries

We only consider finite undirected graphs without loops and multiple edges. Let $G$ be a graph. We denote the vertex set of $G$ by $V_G$ and the edge set by $E_G$. The subgraph of $G$ induced by a subset $U \subseteq V_G$ is denoted by $G[U]$. We say that $U \subseteq V_G$ is *connected* if $G[U]$ is a connected graph. The graph $G-U$ is the graph obtained from $G$ by removing all vertices in $U$. If $U = \{u\}$, we also write $G-u$. The *open neighborhood* of a vertex $u \in V_G$ is defined as $N_G(u) = \{v \mid uv \in E_G\}$, and its *closed neighborhood* is defined as $N_G[u] = N_G(u) \cup \{u\}$. For $U \subseteq V_G$, $N_G[U] = \cup_{u \in U} N_G[u]$. The degree of a vertex $u \in V_G$ is denoted $d_G(u) = |N_G(u)|$. The *distance* $\text{dist}_G(u,v)$ between a pair of vertices $u$ and $v$ of $G$ is the number of edges of a shortest path between them. Two sets $U, U' \subseteq V_G$ are called *adjacent* if there exist vertices $u \in U$ and $u' \in U'$ such that $uu' \in E_G$. A set $U \subseteq V_G$ *dominates* a vertex $w$ if $w \in N_G[U]$, and $U$ *dominates* a set $W \subseteq V_G$ if $U$ dominates each vertex of $W$. In these two cases, we also say that $G[U]$ *dominates* $w$ or $W$, respectively. A set $U \subseteq V_G$ is a *dominating set* of $G$ if $U$ dominates $V_G$.

The graph $P = u_1 \cdots u_k$ denotes the *path* with vertices $u_1, \ldots, u_k$ and edges $u_i u_{i+1}$ for $i = 1, \ldots, k-1$. We also say that $P$ is a *$(u_1, u_k)$-path*. For a path $P$ with some specified end-vertex $s$, we write $x \prec_s y$ if $x \in V_P$ lies in $P$ between $s$ and $y \in V_P$; in this definition, we allow that $x = s$ or $x = y$. A pair of vertices $\{x, y\}$ is a *dominating pair* if the vertex set of every $(x,y)$-path is a dominating set of $G$. Corneil et al. [3, 4] proved the following structural theorem.

**Theorem 1 ([3, 4]).** *Every connected AT-free graph has a dominating pair and such a pair can be found in linear time.*

Using these results, Kloks et al. [12] gave the following tool for constructing dynamic programming algorithms on AT-free graphs. For a vertex $u$ of a graph $G$, we call the sets $L_i(u) = \{v \in V_G \mid \text{dist}_G(u,v) = i\}$ the *BFS-levels* of $G$. Note that the BFS-levels of a vertex can be determined in linear time by the Breadth-First Search algorithm (BFS).

**Theorem 2 ([12]).** *Every connected AT-free graph contains a dominating path $P = u_0 \cdots u_\ell$ that can be found in linear time such that i) $\ell$ is the number of BFS-levels of $u_0$, ii) $u_i \in L_i(u_0)$ for $i = 1, \ldots, \ell$, and iii) each $z \in L_i(u_0)$ is adjacent to $u_{i-1}$ or to $u_i$ for all $1 \leq i \leq \ell$.*

## 3 Set-Restricted Disjoint Paths

We show that Set-Restricted $k$-Disjoint Paths can be solved in polynomial time on AT-free graphs for any fixed integer $k$. We need some extra terminology.

Let $G$ be a graph, and let $W \subseteq V_G$. Consider an induced path $P$ in $G$. Then $V_P \cap W$ and $V_P \setminus W$ induce a collection of subpaths of $P$ called $W$-*segments*, or *segments* if no confusion is possible. Segments induced by $V_P \cap W$ are said to lie *inside* $W$, whereas segments induced by $V_P \setminus W$ lie *outside* $W$. We need the following three lemmas (two proofs are omitted due to space restrictions).

**Lemma 1.** *Let $P$ be an induced path in an AT-free graph $G$. Let $U \subseteq V_G$ be connected. Then $P$ has at most three segments inside $N_G[U]$.*

**Lemma 2.** *Let $P$ be an induced path in an AT-free graph $G$. Let $U \subseteq V_G$ be connected. Then every segment of $P$ outside $N_G[U]$ that contains no end-vertex of $P$ has at most two vertices.*

The next lemma directly follows from the condition on the path $P$ to be induced.

**Lemma 3.** *Let $u$ be a vertex of an induced path $P$ in a graph $G$. Then $P$ has one segment inside $N_G[u]$ and this segment has at most three vertices.*

Let $G$ be a graph with terminal pairs $(s_1, t_1), \ldots (s_k, t_k)$ and corresponding domains $U_1, \ldots, U_k$. Let $\{P_1, \ldots, P_k\}$ be a set of mutually vertex-disjoint paths, such that $P_i$ is a path from $s_i$ to $t_i$ using only vertices from $U_i$ for $i = 1, \ldots, k$. We say that $\{P_1, \ldots, P_k\}$ is a *solution*. A solution $\{P_1, \ldots, P_k\}$ is *minimal* if no $P_i$ can be replaced by a shorter $(s_i, t_i)$-path $P'_i$ that uses only vertices of $U_i$ in such a way that $P_1, \ldots, P_{i-1}, P'_i, P_{i+1}, \ldots, P_k$ are mutually vertex-disjoint. Clearly, every yes-instance of Set-Restricted $k$-Disjoint Paths has a minimal solution. We also observe that any path in a minimal solution is induced. We need Lemma 4 (proof omitted).

**Lemma 4.** *Let $G$ be a graph with terminal pairs $(s_1, t_1), \ldots (s_k, t_k)$ and corresponding domains $U_1, \ldots, U_k$. Let $u \in U_i$ for some $1 \leq i \leq k$, and let $\{P_1, \ldots, P_k\}$ be a minimal solution with $u \notin \bigcup_{j=1}^{k} V_{P_j}$. Then $P_i$ has at most two segments inside $N_G[u]$. Moreover, if $P_i$ has one segment inside $N_G[u]$, then $P_i$ has at most three vertices. If $P_i$ has two segments $Q_1$ and $Q_2$ inside $N_G[u]$, then $Q_1$ and $Q_2$ each has precisely one vertex, and the segment $Q'$ outside $N_G[u]$ that lies between $Q_1$ and $Q_2$ in $P_i$ also has one vertex.*

We apply dynamic programming to prove that Set-Restricted $k$-Disjoint Paths is polynomial-time solvable on AT-free graphs for every fixed integer $k$. Our algorithm solves the decision problem, but can easily be modified to produce the desired paths if they exist. It is based on the following idea. We find a shortest dominating path $u_0 \ldots u_\ell$ in $G$ as described in Theorem 2. For $0 \leq i \leq \ell$, we trace the segments of $(s_j, t_j)$-paths inside $N_G[\{u_0, \ldots, u_i\}]$ by extending the segments inside $N_G[\{u_0, \ldots, u_{i-1}\}]$ in $N_G[u_i] \setminus N_G[\{u_0, \ldots, u_{i-1}\}]$. Note that if some path is traced from the middle, then we have to extend the corresponding segment in two directions, i.e., we have to trace two paths. The paths inside $N_G[u_i] \setminus N_G[\{u_0, \ldots, u_{i-1}\}]$ are constructed recursively, as by Lemmas 3 and 4 we can reduce the number of domains by distinguishing whether $u_i$ is used by one of the paths or not. Hence, it is convenient for us to generalize as follows:

Set-Restricted $r$-Group Disjoint Paths
*Instance:* A graph $H$, positive integers $p_1, \ldots, p_r$, terminal pairs $(s_i^j, t_i^j)$ for $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, p_i\}$, and domains $U_1, \ldots, U_r$.
*Question:* Does $H$ contain mutually vertex-disjoint paths $P_i^j$, where $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, p_i\}$, such that $P_i^j$ is a path from $s_i^j$ to $t_i^j$ using only vertices from $U_i$ for $i = 1, \ldots, r$?

Note that if $p_1 = \ldots = p_r = 1$, then we have Set-Restricted $r$-Disjoint Paths. We say that for each $1 \leq i \leq r$, the pairs $(s_i^1, t_i^1), \ldots, (s_i^{p_i}, t_i^{p_i})$ (or corresponding paths) form a *group*. We are going to solve Set-Restricted $r$-Group Disjoint Paths for induced subgraphs $H$ of $G$ and $r \leq k$ recursively to obtain a solution that can be extended to a solution of Set-Restricted $k$-Disjoint Paths in such a way that $P_i^1, \ldots, P_i^{p_i}$ are disjoint subpaths of the $(s_i, t_i)$-path $P_i$ in the solution of Set-Restricted $k$-Disjoint Paths. Hence, we are interested only in some special solutions of Set-Restricted $r$-Group Disjoint Paths.

For $r = 1$, Set-Restricted $r$-Group Disjoint Paths is the $p_1$-Disjoint Paths problem in $H[U_1]$. By the celebrated result of Robertson and Seymour [16], we immediately get the following lemma.

**Lemma 5.** *For $r = 1$ and any fixed positive integer $p_1$,* Set-Restricted $r$-Group Disjoint Paths *can be solved in $O(n^3)$ time on $n$-vertex graphs.*

Now we are ready to describe our algorithm for Set-Restricted $r$-Group Disjoint Paths. First, we recursively apply the following preprocessing rules.

**Rule 1.** If $H$ has a vertex $u \notin \cup_{i=1}^r U_i$, then we delete it and solve the problem on $H - u$.

**Rule 2.** If there are $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, p_i\}$ such that $s_i^j$ and $t_i^j$ are in different components of $H[U_i]$, then stop and return `No`.

**Rule 3.** If $H$ has components $H_1, \ldots, H_q$ and $q > 1$, then solve the problem for each component $H_h$ for the pairs of terminals $(s_i^j, t_i^j)$ such that $s_i^j, t_i^j \in V_{H_h}$ and the corresponding domains. We return `Yes` if we get a solution for each component $H_h$, and we return `No` otherwise.

**Rule 4.** If $r = 1$, then solve the problem by Lemma 5.

From now we assume that $r \geq 2$ and $H$ is connected. Let $p = p_1 + \ldots + p_r$.

By Theorem 2, we can find a vertex $u_0 \in V_H$ and a dominating path $P = u_0 \ldots u_\ell$ in $H$ with the property that for $i \in \{1, \ldots, \ell\}$, $u_i \in L_i$ and for any $z \in L_i$, $z$ is adjacent to $u_{i-1}$ or $u_i$, where $L_0, \ldots L_\ell$ are the BFS-levels of $u_0$. For $i \in \{0, \ldots, \ell\}$, let $W_i = N_H[\{u_0, \ldots, u_i\}]$, $W_{-1} = \emptyset$, and $S_i = N_G[u_i] \setminus W_{i-1}$. To simplify notations, we assume that for $i > \ell$, $S_i = \emptyset$, and $S_{-1} = \emptyset$. Notice that by the choice of $P$, there are no edges $xy \in E_H$ with $x \in S_j$ and $y \in N_H[\{u_0, \ldots, u_i\}]$ if $j - i > 2$.

Our dynamic programming algorithm keeps a table for each $i \in \{0, \ldots, \ell\}$, $X_i \subseteq S_{i+1}$ and $Y_i \subseteq S_{i+2}$, where $|X_i| \leq 4p, |Y_i| \leq 4p$, and an integer $next_i \in \{0, \ldots, r\}$. The table stores information about segments of $(s_j^h, t_j^h)$-paths inside $W_i$. Recall that each path can have more than one segment inside $W_i$, but in this case by Lemma 1, there are at most three such segments, and by Lemma 2, the number of vertices of the segments outside $W_i$, that join the segments inside, is bounded. We keep information about these vertices in $X_i, Y_i$. If $next_i = 0$, then no path in the partial solution includes $u_{i+1}$, and if $next_i = j > 0$, then only $(s_j^h, \tilde{t}_j^h)$, $(\tilde{s}_j^h, t_j^h)$, $(\tilde{s}_j^h, \tilde{t}_j^h)$-paths can use $u_{i+1}$ (if $i = \ell$, then we assume that $next_i = 0$). For each $i, X_i, Y_i, next_i$, the table stores a collection of records $\mathcal{R}(i, X_i, Y_i, next_i)$ with the elements

$$\{(State_j^h, R_j^h) | 1 \leq j \leq r, 1 \leq h \leq p_i\},$$

where $R_j^h$ are ordered multisets of size at most two without common vertices except (possibly) terminals $s_1, \ldots, s_k, t_1, \ldots, t_k$ of the original instance of Set-Restricted $k$-Disjoint Paths, $R_j^h \subseteq U_j$, and where each $State_j^h$ can have one of the following five values:

*Not initialized*, *Started from s*, *Started from t*, *Started from middle*, *Completed.*

These records correspond to a partial solution of Set-Restricted $r$-Group Disjoint Paths for $H_i = H[W_i \cup X_i \cup Y_i]$ with the following properties.

- If $State_j^h =$ *Not initialized*, then $(s_j^h, t_j^h)$-paths have no vertices in $H_i$ in the partial solution and $R_j^h = \emptyset$.
- If $State_j^h =$ *Started from s*, then $s_j^h \in W_i$, $t_j^h \notin V_{H_i}$ and $R_j^h$ contains one vertex. Let $R_j = (\tilde{t}_j^h)$. Then $\tilde{t}_j^h \in S_{i-1} \cup S_i$ and the partial solution contains an $(s_j^h, \tilde{t}_j^h)$-path.
- If $State_j^h =$ *Started from t*, then $s_j^h \notin V_{H_i}$, $t_j^h \in W_i$ and $R_j^h$ contains one vertex. Let $R_j^h = (\tilde{s}_j^h)$. Then $\tilde{s}_j^h \in S_{i-1} \cup S_i$ and the partial solution contains an $(\tilde{s}_j^h, t_j^h)$-path.
- If $State_j^h =$ *Started from middle*, then $s_j^h, t_j^h \notin V_{H_i}$ and $R_j^h$ contains two vertices. Let $R_j^h = (\tilde{s}_j^h, \tilde{t}_j^h)$ (it can happen that $\tilde{t}_j^h = \tilde{s}_j^h$ ). Then $\tilde{s}_j^h, \tilde{t}_j^h \in S_{i-1} \cup S_i$ and the partial solution contains an $(\tilde{s}_j^h, \tilde{t}_j^h)$-path.
- If $State_j^h =$ *Completed*, then $s_j^h, t_j^h \in W_i$, $R_j^h = \emptyset$, and it is assumed that the partial solution contains an $(s_j^h, t_j^h)$-path.

We consequently construct the tables for $i = 0, \ldots, \ell$. The algorithm returns `Yes` if $\mathcal{R}(\ell, X_\ell, Y_\ell, next_\ell)$ for $X_\ell = Y_\ell = \emptyset$ contains the record $\{(State_j^h, R_j^h) | 1 \leq j \leq r, 1 \leq h \leq p_i\}$, where each $State_j^h =$(Completed). The details and the proof of the main theorem have been omitted.

**Theorem 3.** Set-Restricted $k$-Disjoint Paths *can be solved in* $O(n^{f(k)})$ *time for* $n$*-vertex AT-free graphs for some function* $f(k)$ *that only depends on* $k$*.*

## 4 Induced Minors

In this section we consider the $H$-Induced Minor problem. It is convenient for us to represent this problem in the following way. An $H$*-witness structure* of $G$ is a collection of $|V_H|$ non-empty mutually disjoint sets $W(x) \subseteq V_G$, one set for each $x \in V_H$, called $H$*-witness sets*, such that

(i) each $W(x)$ is a connected set; and
(ii) for all $x, y \in V_H$ with $x \neq y$, sets $W(x)$ and $W(y)$ are adjacent in $G$ if and only if $x$ and $y$ are adjacent in $H$.

Observe that $H$ is an induced minor of $G$ if and only if $G$ has an $H$-witness structure.

**Theorem 4.** *$H$-*Induced Minor *can be solved in polynomial time on AT-free graphs for any fixed graph $H$.*

*Proof.* Suppose that $H$ is an induced minor of $G$. Then $G$ has an $H$-witness structure, i.e., sets $W(x) \subseteq V_G$ for $x \in V_H$. For each $x \in V_H$, $G[W(x)]$ is a connected AT-free graph. Hence, by Theorem 1, $G[W(x)]$ has a dominating pair $(u_x, v_x)$.

For each $x \in V_H$, we guess the pair $(u_x, v_x)$ (it can happen that $u_x = v_x$), and guess at most six vertices of a shortest $(u_x, v_x)$-path $P_x$ in $G[W(x)]$ as follows: if $P_x$ has at most five vertices, then we guess all vertices of $P_x$, and if $P_x$ has at least six vertices, then we guess the first three vertices $u_1^x, u_2^x, u_3^x$ and the last three vertices $v_1^x, v_2^x, v_3^x$ such that $u_x = u_1^x$, $v_x = v_3^x$ and $u_1^x \prec_{u_x} u_2^x \prec_{u_x} u_3^x \prec_{u_x} v_1^x \prec_{u_x} v_2^x \prec_{u_x} v_3^x$ in $P_x$. Observe that $P_x$ is an induced path. We denote by $X_1, X_2$ the partition of $V_H$ (one of the sets can be empty), where for $x \in X_1$, all at most five vertices of $P_x$ were chosen, and for $x \in X_2$, we have the vertices $u_1^x, u_2^x, u_3^x, v_1^x, v_2^x, v_3^x$. Further, for each edge $xy \in E_H$, we guess adjacent vertices $s_{xy}, s_{yx} \in V_G$, where $s_{xy} \in W(x)$ and $s_{yx} \in W(y)$. Notice that the vertices $s_{xy}$ are not necessarily distinct, and some of them can coincide with the vertices chosen to represent $P_x$. Let $S(x) = \{s_{xy} | xy \in E_H\}$. All the guesses should be consistent with the witness structure, i.e., vertices included in distinct $W(x)$ should be distinct, and if $xy \notin E_H$, then the vertices included in $W(x)$ and $W(y)$ should be non-adjacent in $G$.

For $x \in X_1$, we check whether the guessed path $P_x$ dominates $S(x)$, and if it is so, then we let $W'(x) = V_{P_x} \cup S(x)$. Otherwise we discard our choice.

Recall that we already selected some vertices, and that we cannot use these vertices and also not their neighbors in case non-adjacencies in $H$ forbid this. Hence, for each $x \in X_2$, we obtain the set

$$\begin{aligned} U_x = V_G \setminus \big( & (\cup_{y \in X_1, xy \in E_H} W'(y)) \cup (\cup_{y \in X_1, xy \notin E_H} N_G[W'(y)]) \cup \\ & \cup (\cup_{y \in X_2, xy \in E_H} (S(y) \cup \{u_1^y, u_2^y, u_3^y, v_1^y, v_2^y, v_3^y\}) \cup \\ & \cup (\cup_{y \in X_2 \setminus \{x\}, xy \notin E_H} N_G[S(y) \cup \{u_1^y, u_2^y, u_3^y, v_1^y, v_2^y, v_3^y\}] \cup \\ & \cup N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}] \big) \cup \{u_1^x, u_2^x, u_3^x, v_1^x, v_2^x, v_3^x\}. \end{aligned}$$

Then for each $x \in X_2$, we check whether $S'(x) = S(x) \setminus N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]$ is included in one component of $G[U_x]$. If it is not so, then we discard our choice, since we cannot have a path with the first vertices $u_1^x, u_2^x, u_3^x$ and the last vertices $v_1^x, v_2^x, v_3^x$ that dominates $S(x)$. Otherwise we denote by $U'_x$ the set of vertices of the component of $G[U_x]$ that contains $S'(x)$. Notice that $(u_1^x, v_3^x)$ is a dominating pair in $G[U'_x]$ for $x \in X_2$. To show it, consider a dominating pair $(u, v)$ in $G[U'_x]$.

Any $(u, v)$-path $P$ dominates $u_1^x$ and $v_3^x$. It follows that one vertex of the pair is in $\{u_1^x, u_2^x\}$ and another is in $\{v_2^x, v_3^x\}$. It remains to observe that if $u_2^x$ ($v_2^x$ respectively) is in the pair, then it can be replaced by $u_1^x$ ($v_3^x$ respectively). We solve Set-Restricted $|X_2|$-Disjoint Paths for the pairs of terminals $(u_1^x, v_3^x)$ with domains $U'_x$ for $x \in X_2$. If we get a No-answer, then we discard our guess since there are no $P_x$ that satisfy our choices. Otherwise, let $P'_x$ be the $(u_1^x, v_3^x)$-path in the obtained solution for $x \in X_2$. We let $W'(x) = P'_x \cup S(x)$.

We claim that the sets $W'(x)$ compose an $H$-witness structure. To show it, observe first that by the construction of these sets, $W'(x)$ are disjoint. If $xy \in E_H$, then as $s_{xy} \in W'(x)$ and $s_{yx} \in W'(y)$, $W'(x)$ and $W'(y)$ are adjacent. It remains to prove that if $xy \notin E_G$, then $W'(x)$ and $W'(y)$ are not adjacent. To obtain a contradiction, assume that $W'(x)$ and $W'(y)$ are adjacent for some $x, y \in V_H$, i.e., there is $uv \in E_G$ with $u \in W'(x)$ and $v \in W'(y)$, where $xy \notin E_H$. By the construction of $W'(x), W'(y)$, $x, y \in X_2$. Moreover, $u \notin N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]$ or $v \notin N_G[\{u_1^y, u_2^y, v_2^y, v_3^y\}]$. If $u \notin N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]$, then we consider $u_1^x, v_3^x, u_1^y$ and observe that these vertices compose an asteroidal triple. Clearly, the $(u_1^x, v_3^x)$-path $P'_x$ avoids $N_G[u_1^y]$, because $N_G[u_1^y] \cap U_x = \emptyset$. Because $u \notin N_G[\{u_1^x, u_2^x, v_2^x, v_3^x\}]$, $u$ is either in $P'_x$ or adjacent to a vertex in $P'_x$ and $v$ is either in $P'_y$ or adjacent to a vertex in $P'_y$, $G[W'(x) \cup W'[y]] - N_G[u_1^x]$ and $G[W'(x) \cup W'[y]] - N_G[v_3^x]$ are connected. Hence, there are $(u_1^x, u_1^y)$ and $(v_3^x, u_1^y)$-paths that avoid $N_G[v_3^x]$ and $N_G[u_1^x]$ respectively. By symmetry, we conclude that if $v \notin N_G[\{u_1^y, u_2^y, v_2^y, v_3^y\}]$, then $u_1^y, v_3^y, u_1^x$ is an asteroidal triple. This contradiction proves our claim. To complete the proof, note that we guess at most $6|V_H| + 2|E_H|$ vertices of $G$, and we can consider all possible choices in time $n^{O(|V_H|+|E_H|)}$, where $n = |V_G|$. If for one of the choices we get an $H$-witness structure, then $H$ is an induced minor of $G$, otherwise we return No. As we can solve Set-Restricted $|X_2|$-Disjoint Paths in time $n^{f(|V_H|)}$ by Theorem 3, the claim follows. □

A graph is *cobipartite* if its vertex set can be partitioned into two cliques. Such a graph is AT-free. Hence, the next theorem complements Theorem 4. It is proven by a reduction from the Clique problem; the details have been omitted.

**Theorem 5.** *The $H$-Induced Minor problem is NP-complete for cobipartite graphs, and W[1]-hard for cobipartite graphs when parameterized by $|V_H|$.*

## 5 Concluding Remarks

We have presented a polynomial-time algorithm that solves Set-Restricted $k$-Disjoint Paths on AT-free graphs for any fixed integer $k$, and applied this algorithm to solve $H$-Induced Minor in polynomial time on this graph class for any fixed graph $H$. We give (without proofs) two further applications of our algorithm for Set-Restricted $k$-Disjoint Paths.

**Theorem 6.** Set-Restricted $k$-Disjoint Connected Subgraphs *can be solved in polynomial time on AT-free graphs for any fixed integer $k$.*

**Theorem 7.** *$H$-Contractibility can be solved in polynomial time on AT-free graphs for any fixed triangle-free graph $H$.*

The *join* of two vertex-disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \bowtie G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{uv \mid u \in V_1, v \in V_2\})$. A graph $G$ contains a graph $H$ as an induced minor if and only if $K_1 \bowtie G$ contains $K_1 \bowtie H$ as a contraction [10]. This fact together with Theorem 5 yields Corollary 1.

**Corollary 1.** *$H$-Contractibility is* NP*-complete for cobipartite graphs, and* W[1]*-hard for cobipartite graphs when parameterized by $|V_H|$.*

Determining the complexity classification of $H$-Contractibility on AT-free graphs when $H$ is a fixed graph that is not triangle-free is an open problem.

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