Tight Complexity Bounds for FPT Subgraph Problems Parameterized by Clique-width[☆]

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Abstract

We give tight algorithmic lower and upper bounds for some double-parameterized subgraph problems when the clique-width of the input graph is one of the parameters. Let G be an arbitrary input graph on n vertices with clique-width at most w. We prove the following results.

- The Dense (Sparse) k-Subgraph problem, which asks whether there exists an induced subgraph of G with k vertices and at least q edges (at most q edges, respectively), can be solved in time $k^{O(w)} \cdot n$, but it cannot be solved in time $2^{o(w \log k)} \cdot n^{O(1)}$ unless the Exponential Time Hypothesis (ETH) fails.
- The d-Regular Induced Subgraph problem, which asks whether there exists a d-regular induced subgraph of G, and the Minimum Subgraph of Minimum Degree at least d problem, which asks whether there exists a subgraph of G with k vertices and minimum degree at least d, can be solved in time $d^{O(w)} \cdot n$, but they cannot be solved in time $2^{o(w \log d)} \cdot n^{O(1)}$ unless ETH fails.

Keywords: Parameterized complexity, Clique-width, Exponential Time Hypothesis

1. Introduction

The notion of clique-width introduced by Courcelle and Olariu [14] (we refer the reader to the survey [24] for further information on different width parame-

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ters) has now become one of the fundamental parameters in Graph Algorithms. Many problems which are hard on general graphs can be solved efficiently when the input is restricted to graphs of bounded clique-width. The meta-theorem of Courcelle, Makowsky, and Rotics [13] states that all problems expressible in MS_1-logic are *fixed parameter tractable* (FPT), when parameterized by the clique-width of the input graph (see the books of Downey and Fellows [18] and Flum and Grohe [21] for a detailed treatment of parameterized complexity). In other words, this theorem shows that any problem expressible in MS_1-logic can be solved for graphs of clique-width at most w in time $f(w) \cdot |I|^{O(1)}$, where $|I|$ is the size of the input and f is a computable function depending on the parameter w only. Here, the superexponential function f is defined by a logic formula, and it grows very fast.

The basic method for constructing algorithms for graphs of bounded clique-width is to use dynamic programming along an expression tree (the definition is given in Section 2). Computing clique-width is an NP-hard problem [20], but it can be approximated and a corresponding expression tree can be constructed in FPT-time [23, 32]. In our paper it is always assumed that an expression tree is given. In this case dynamic programming algorithms can be relatively efficient: usually single-exponential in the clique-width. A natural question to ask is whether the running times of such algorithms are asymptotically optimal up to some reasonable complexity conjectures.

The *Exponential Time Hypothesis (ETH)* [25] asserts that there does not exist an algorithm for solving 3-SAT running in time $2^{o(n)}$ on a formula with n variables; this is equivalent to the parameterized complexity conjecture that FPT $\neq$ M[1] [17, 21]. The Exponential Time Hypothesis has proved to be an effective tool for establishing tight complexity bounds for parameterized problems. We refer the reader to the survey by Lokshtanov, Marx and Saurabh [28] for an overview of the field and mention here only some results related to our paper. Chen et al. [8, 9, 10] showed that there is no algorithm for k-Clique running in time $f(k)n^{o(k)}$, for n-vertex graphs, unless ETH fails (on the other hand it is easily seen that k-Clique can be solved in time $n^{O(k)}$). The lower bound on the k-Clique problem can be extended to some other parameterized problems via linear FPT-reductions [9, 10]. In particular, for problems parameterized by clique-width, Fomin et al. [22] proved that Max-Cut and Edge Dominating Set cannot be solved in time $f(w)n^{o(w)}$ on n-vertex graphs of clique-width at most w, unless ETH collapses. Lokshtanov, Marx and Saurabh [29] considered several FPT problems solvable in time $2^{O(k \log k)} \cdot |I|^{O(1)}$ and showed that a $2^{o(k \log k)} \cdot |I|^{O(1)}$-time algorithm for these problems would violate ETH. To do this, they introduced special restricted versions of some basic problems like k-Clique on graphs with k^2 vertices (and with some other restrictions) and proved that these problems cannot be solved in time $2^{o(k \log k)} \cdot k^{O(1)}$ unless ETH collapses. These results open the possibility of establishing algorithmic lower bounds for natural problems. We use this approach to prove asymptotically tight bounds for some double-parameterized subgraph problems when the clique-width of the input graph is one of the parameters. These results give the first known bounds for such types of problems parameterized by clique-width.

First, we consider the DENSE k-SUBGRAPH problem (also known as the k-CLUSTER problem). This problem asks whether, given a graph G and positive integers k and q, there exists an induced subgraph of G with k vertices and at least q edges. Clearly, DENSE k-SUBGRAPH is NP-hard since it is a generalization of the k-CLIQUE problem. It remains NP-hard, even when restricted to comparability graphs, bipartite graphs and chordal graphs [12], as well as on planar graphs [26]. Polynomial algorithms were given for cographs, split graphs [12], and for graphs of bounded tree-width [26]. Considerable work has been done on approximation algorithms for this problem [3, 4, 15, 19, 27].

Next, we consider some degree-constrained subgraph problems. The objective in such problems is to find a subgraph satisfying certain lower or upper bounds on the degree of each vertex. Typically it is necessary to either check the existence of a subgraph satisfying the degree constraints or to minimize (maximize) some parameter (usually the size of the subgraph).

The d-REGULAR SUBGRAPH problem, which asks whether a given graph contains a d-regular subgraph, has been intensively studied. We mention here only some complexity results. Chvátal et al. [11] proved that this problem is NP-complete for $d = 3$. It was shown that the problem with $d = 3$ remains NP-complete for planar bipartite graphs with maximum degree four, and that when $d \geq 3$, it is NP-complete even for bipartite graphs with maximum degree at most $d + 1$. Some further results were given in [7, 33, 34, 35]. We consider a variant of this problem called d-REGULAR INDUCED SUBGRAPH, where we ask whether a given graph G contains a d-regular *induced* subgraph. This variant of the problem has also been studied. In particular, the parameterized complexity of different variants of the problem was considered by Moser and Thilikos [31] and by Mathieson and Szeider [30]. Observe that, trivially, d-REGULAR INDUCED SUBGRAPH can be solved in polynomial time for $d \leq 2$, and it easily follows from the known hardness results for d-REGULAR SUBGRAPH that d-REGULAR INDUCED SUBGRAPH is NP-complete for any fixed $d \geq 3$.

In [2] Amini et al. introduced the MINIMUM SUBGRAPH OF MINIMUM DEGREE AT LEAST d problem. This problem asks whether, given a graph G and positive integers d and k, there exists a subgraph of G with at most k vertices and minimum degree at least d. The parameterized complexity of the problem was considered in [2]. Some other hardness and approximation results can be found in [1].

Our main results and the organization of the paper.. In Section 2 we give some basic definitions and some preliminary results. In Section 3 we consider the DENSE k-SUBGRAPH and SPARSE k-SUBGRAPH problems. The SPARSE k-SUBGRAPH problem is dual to DENSE k-SUBGRAPH and it asks whether, given a graph G and positive integers k and q, there exists an induced subgraph of G with k vertices and at most q edges. We prove that these problems can be solved in time $k^{O(w)} \cdot n$ for n-vertex graphs of clique-width at most w if an expression tree of width w is given, but they cannot be solved in time $2^{o(w \log k)} \cdot n^{O(1)}$ unless ETH fails even if an expression tree of width w is included in the input. In Section 4 and 5 we consider the d-REGULAR INDUCED SUBGRAPH and MINI-

mum Subgraph of Minimum Degree at least d problems respectively. We construct dynamic programming algorithms which solve these problems in time $d^{O(w)} \cdot n$ for n-vertex graphs of clique-width at most w if an expression tree of width w is given, and then prove that these problems cannot be solved in time $2^{o(w \log d)} \cdot n^{O(1)}$ unless ETH fails even if an expression tree of width w is provided. We conclude the paper with some open problems.

2. Definitions and Preliminary Results

Graphs.. We consider finite undirected graphs without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$. A set $S \subseteq V(G)$ of pairwise adjacent vertices is called a *clique*. For $v \in V(G)$, $E_G(v)$ denotes the set of edges incident with v. The *degree* of a vertex v is denoted by $d_G(v)$. For a non-negative integer d, a graph G is called *d-regular* if all vertices of G have degree d. For a graph G, the *incidence graph* of G is the bipartite graph $I(G)$ with vertex set $V(G) \cup E(G)$ such that $v \in V(G)$ and $e \in E(G)$ are adjacent if and only if v is incident with e in G. We denote by $\overline{G}$ the *complement* of a graph G, i.e. the graph with vertex set $V(G)$ such that any two distinct vertices are adjacent in $\overline{G}$ if and only if they are non-adjacent in G. For a set of vertices $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S, and by $G - S$ we denote the graph obtained from G by the removal of all the vertices of S, i.e. the subgraph of G induced by $V(G) \setminus S$.

Clique-width.. Let G be a graph, and let w be a positive integer. A w-graph is a graph whose vertices are labeled by integers from $\{1, 2, \ldots, w\}$. We call the w-graph consisting of exactly one vertex v labeled by some integer i from $\{1, 2, \ldots, w\}$ an initial w-graph. The *clique-width* $\mathbf{cwd}(G)$ is the smallest integer w such that G can be constructed by means of repeated application of the following four operations: (1) *introduce*: construction of an initial w-graph with vertex v labeled by i (denoted by $i(v)$), (2) *disjoint union* (denoted by $\oplus$), (3) *relabel*: changing the labels of each vertex labeled i to j (denoted by $\rho_{i\to j}$) and (4) *join*: joining all vertices labeled by i to all vertices labeled by j by edges (denoted by $\eta_{i,j}$).

An *expression tree* of a graph G is a rooted tree T of the following form.

- The nodes of T are of four types: i, $\oplus$, η and ρ.
- Introduce nodes $i(v)$ are leaves of T, and they correspond to initial w-graphs with vertices v, which are labeled i.
- A union node $\oplus$ stands for a disjoint union of graphs associated with its children.
- A relabel node $\rho_{i\to j}$ has one child and is associated with the w-graph resulting from the relabeling operation $\rho_{i\to j}$ applied to the graph corresponding to the child.

- A join node $\eta_{i,j}$ has one child and is associated with the w-graph resulting from the join operation $\eta_{i,j}$ applied to the graph corresponding to the child.
- The graph G is isomorphic to the graph associated with the root of T (with all labels removed).

The *width* of the tree T is the number of different labels appearing in T. If a graph G has $\mathbf{cwd}(G) \leq w$ then it is possible to construct a rooted expression tree T of G with *width* w. Given a node X of an expression tree, the graph G_X is the graph formed by the subtree of the expression tree rooted at X.

Parameterized reductions.. We refer the reader to the books [18, 21] for a detailed treatment of parameterized complexity. Here we only define the notion of parameterized (linear) reduction, which is the main tool for establishing our results. For parameterized problems A, B, we say that A is (uniformly many:1) FPT-*reducible* to B if there exist functions $f, g : \mathbb{N} \to \mathbb{N}$, a constant $\alpha \in \mathbb{N}$ and an algorithm Φ which transforms an instance (x, k) of A into an instance $(x', g(k))$ of B in time $f(k)|x|^\alpha$ so that $(x, k) \in A$ if and only if $(x', g(k)) \in B$. The reduction is called *linear* if $g(k) = O(k)$.

Capacitated domination.. For our reductions we use a variant of the Capacitated Dominating Set problem. The parameterized complexity of this problem, with the tree-width of the input graph being the parameter, was considered in [5, 16].

A *red-blue capacitated graph* is a pair (G, c), where G is a bipartite graph with a vertex bipartition into sets R and B, and $c\colon R \to \mathbb{N}$ is a *capacity* function such that $1 \leq c(v) \leq d_G(v)$ for every vertex $v \in R$. The vertices of the set R are called *red* and the vertices of B are called *blue*. A set $S \subseteq R$ is called a *capacitated dominating set* if there is a *domination mapping* $f\colon B \to S$ which maps every vertex in B to one of its neighbors such that the total number of vertices mapped by f to any vertex $v \in S$ does not exceed its capacity $c(v)$. We say that for a vertex $v \in S$, vertices in the set $f^{-1}(v)$ are *dominated by* v. The Red-Blue Capacitated Dominating Set (or Red-Blue CDS) problem asks whether, given a red-blue capacitated graph (G, c) and a positive integer k, there exists a capacitated dominating set S for G containing at most k vertices. A capacitated dominating set $S \subseteq R$ is called *saturated* if there is a domination mapping f which saturates all vertices of S, that is, $|f^{-1}(v)| = c(v)$ for each $v \in S$. The Red-Blue Exact Saturated Dominating Set problem (Red-Blue Exact Saturated CDS) takes a red-blue capacitated graph (G, c) and a positive integer k as an input and asks whether there exists a saturated capacitated dominating set with exactly k vertices.

The next proposition immediately follows from the results proved in [22].

Proposition 1. *The* Red-Blue CDS *and* Red-Blue Exact Saturated CDS *problems cannot be solved in time $f(w) \cdot n^{o(w)}$, where n is the number of vertices of the input graph G and w is the clique-width of the incidence graph $I(G)$, unless ETH fails, even if an expression tree of width w for $I(G)$ is given.*

The proof of Proposition 1 uses the result of Chen et al. [8, 9, 10] that there is no algorithm for k-Clique (finding a clique of size k) running in time $f(k) \cdot n^{o(k)}$ unless there exists an algorithm for solving 3-SAT running in time $2^{o(n)}$ on a formula with n variables. Proposition 1 was proved via a linear reduction from the k-Multi-Colored Clique problem (see [5, 22]). The k-Multi-Colored Clique problem asks for a given k-partite graph $G = (V_1 \cup \cdots \cup V_k, E)$, where $V_1, \ldots, V_k$ are sets of the k-partition, whether there is a k-clique in G. It should be noted that the construction of an expression tree of bounded width is part of the reduction and it is done in polynomial time. Lokshtanov, Marx and Saurabh [29] considered a special restricted variant of k-Multi-Colored Clique called $k \times k$-Clique. In this variant of the problem $|V_1| = \ldots = |V_k| = k$. They proved the following.

Proposition 2 ([29])**.** *The $k \times k$-Clique problem cannot be solved in time $2^{o(k \log k)} \cdot n^{O(1)}$, where n is the number of vertices of the input graph G, unless ETH fails.*

By replacing k-Multi-Colored Clique by the $k \times k$-Clique problem in the reductions used for the proof of Proposition 1, we obtain the following corollary.

Corollary 1. *The* Red-Blue CDS *and* Red-Blue Exact Saturated CDS *problems cannot be solved in time $2^{o(w \log n)} \cdot n^{O(1)}$, where n is the number of vertices of the input graph G and w is the clique-width of the incidence graph $I(G)$, unless ETH fails, even if an expression tree of width w for $I(G)$ is given.*

Observe that Corollary 1 gives a slightly stronger claim than Proposition 1: while $o(w) \cdot \log n = o(w \log n)$, it is not so the other way around.

3. Sparse and Dense k-Subgraph problems

In this section we consider the Dense k-Subgraph and Sparse k-Subgraph problems. The aim of this section is the proof of the following theorem.

Theorem 1. *The* Sparse k-Subgraph *problem can be solved in time $k^{O(w)} \cdot n$ on n-vertex graphs of clique-width at most w if an expression tree of width w is given, but it cannot be solved in time $2^{o(w \log k)} \cdot n^{O(1)}$ unless ETH fails, even if an expression tree of width w is given.*

Clearly, Sparse k-Subgraph and Dense k-Subgraph are dual, i.e. Sparse k-Subgraph is equivalent to Dense k-Subgraph for the complement of the input graph. Since for any graph G, $\mathbf{cwd}(\overline{G}) \leq 2 \cdot \mathbf{cwd}(G)$ (see e.g. [14, 36]), we can immediately get the following corollary.

Corollary 2. *The* Dense k-Subgraph *problem can be solved in time $k^{O(w)} \cdot n$ on n-vertex graphs of clique-width at most w if an expression tree of width w is given, but it cannot be solved in time $2^{o(w \log k)} \cdot n^{O(1)}$ unless ETH fails, even if an expression tree of width w is given.*

3.1. Algorithmic upper bounds for Sparse k-Subgraph

We sketch a dynamic programming algorithm for solving SPARSE k-SUBGRAPH in time $k^{O(w)} \cdot n$ on graphs of clique-width at most w. The algorithm is based on the following observation.

Let G be a graph with n vertices, and let T be an expression tree for G of width w. For a node X of T, let $U_1(X), \ldots, U_w(X)$ be the sets of vertices of G_X labeled $1, \ldots, w$, respectively. Recall that for a graph G, we are looking for an induced subgraph H on k vertices with at most q edges. For a node X of T, denote by H_X the subgraph of G_X induced by $V(H) \cap V(G_X)$ (it can happen that $V(H) \cap V(G_X) = \emptyset$). If X is an introduce node, then H_X has no edges. If X is a relabel node, then the number of edges of H_X is the same as the the number of edges of H_Y for the unique child Y of X. If X is a union node, then the number of edges of H_X is the sum of the numbers of edges of H_Y and H_Z where Y and Z are the children of X. Finally, if X is a join node $\eta_{i,j}$, then $|E(H_X)| = |E_Y| + s_i s_j$ where Y is the unique child of X and s_i, s_j are the number of vertices of H_X labeled i and j respectively in G_X. It indicates that each partial solution for a node X can be encoded as the minimum number of edges of an induced subgraph of G_X that has exactly s_i vertices of $U_i(X)$ for $i \in \{1, \ldots, w\}$.

Now we describe formally what we store in the tables corresponding to the nodes in an expression tree. The table of data for the node X contains entries which store a positive integer $p \leq q$ and a vector $(s_1, \ldots, s_w)$ of non-negative integers such that $s = s_1 + \ldots + s_w \leq k$ for $i \in \{1, \ldots, w\}$, for which p is the minimum number of edges of an induced subgraph H with s vertices such that for $i \in \{1, \ldots, w\}$, $s_i = |U_i(X) \cap V(H)|$. If X is the root node of T then G contains an induced subgraph with k vertices and at most q edges if and only if the table for X contains an entry with the parameter $p \leq q$ and vector $(s_1, \ldots, s_w)$ such that $s_1 + \ldots + s_w = k$.

We create and update the tables as follows.

Introduce Node: Tables for introduce nodes of T are constructed in a straightforward manner. Suppose that $X = i(v)$ for $v \in V(G)$ and $i \in \{1, \ldots, w\}$. Then the table of data for the node X contains two entries with $p = 0$ and the vectors $(s_1, \ldots, s_w)$ such that $s_j = 0$ for $j \in \{1, \ldots, w\}$, $j \neq i$, $s_i = 0$ for the first entry and $s_i = 1$ for the second.

Relabel Node: Suppose that X is a relabel node $\rho_{i \to j}$, and Y is a child of X. Then the table for X contains an entry with an integer p and a vector $(s_1, \ldots, s_w)$ if and only if $s_i = 0$ and p is the minimum integer for which the table for Y contains the entry with p and the vector $(s'_1, \ldots, s'_w)$ such that

- $s_l = s'_l$ for $l \in \{1, \ldots, w\}$, $l \neq i, j$,
- $s_j = s'_i + s'_j$.

Union Node: Let X be a union node with children Y and Z. In this case the table for X contains an entry with p and a vector $(s_1, \ldots, s_w)$ if and only

if p is the minimum integer for which the tables for Y and Z have the entries p', $(s'_1, \ldots, s'_w)$ and p'', $(s''_1, \ldots, s''_w)$ respectively, such that

- $p = p' + p''$,
- $s_i = s'_i + s''_i$ for $i \in \{1, \ldots, w\}$.

Join Node: Finally, suppose that X is a join node $\eta_{i,j}$ with a child Y. It can be noted that the table for X has an entry p, $(s_1, \ldots, s_w)$ if and only if p is the minimum integer for which the table for Y includes the entry p', $(s_1, \ldots, s_w)$ where $p = p' + s_i s_j$.

Correctness of the algorithm follows from the description of the procedure. Since for each X, the table for X contains at most $(k+1)^w$ vectors and for each vector only one value of the parameter p is stored, the algorithm runs in time $k^{O(w)} \cdot n$. This proves that Sparse k-Subgraph can be solved in time $k^{O(w)} \cdot n$ on graphs of clique-width at most w.

3.2. Lower bounds

To prove our lower bounds we give a reduction from the Red-Blue CDS problem parameterized by the clique-width of the incidence graph of the input graph.

Construction.. Let (G, c, k) be an instance of Red-Blue CDS with $R = \{u_1, \ldots, u_n\}$ being the set of red vertices and $B = \{v_1, \ldots, v_r\}$ being the set of blue vertices. Let m be the number of edges of G. We assume without loss of generality that G has no isolated vertices. Hence, $m \geq n, r$.

First, we construct the auxiliary gadget $F(l)$ for a positive integer l.

Auxiliary gadget $F(l)$***:*** Construct an $l+m+1$-partite graph $K_{2,\ldots,2}$ and denote by x_{i1}, x_{i2} the vertices of the i-th set of the partition (see Figure 1).

We need a technical lemma about properties of the gadget $F(l)$.

Lemma 1. *Let J be a disjoint union of s copies $F_1, \ldots, F_s$ of $F(l_1), \ldots, F(l_s)$, respectively for $l_1, \ldots, l_s \geq 1$, and let $U \subseteq V(J)$.*

- *If $|U| = 2s(m+1)$ then $|E(G[U])| \geq 2sm(m+1)$, and $|E(G[U])| = 2sm(m+1)$ if and only if U contains vertices of $m+1$ sets of the (l_i+m+1)-partition of each copy of $F(l_i)$.*
- *If $|U| < 2s(m+1)$ then $2sm(m+1) - |E(G[U])| \leq (2s(m+1) - |U|) \cdot 2m$.*
- *If $|U| > 2s(m+1)$ then $|E(G[U])| - 2sm(m+1) \geq (|U| - 2s(m+1)) \cdot 2(m+1)$.*

Proof. For $s = 1$, the first claim follows immediately from the definition of $F(l)$. Let $s > 1$. Suppose that $U \subseteq V(J)$ contains $2s(m+1)$ vertices and the number of edges of $J[U]$ is minimum. Let $p_i = V(F_i) \cap U$ for $i \in \{1, \ldots, s\}$. Since U induces a subgraph with the minimum number of edges, U contains vertices of $\lfloor p_i/2 \rfloor$ sets of the $(l_i + m + 1)$-partition of each F_i. Assume that there are $i, j \in \{1, \ldots, s\}$ such that $p_i > 2(m+1)$ and $p_j < 2(m+1)$. Observe that there is a vertex

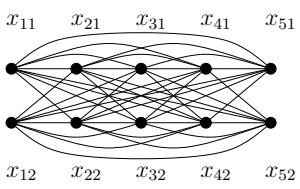
$u \in V(F_i) \cap U$ which is adjacent to at least $2(m+1)$ vertices of U and there is a vertex $v \in V(F_j) \setminus U$ which is adjacent to at most $2m$ vertices of U. Consider the set $U' = (U \setminus \{u\}) \cup \{v\}$. Clearly, $|E(J[U'])| \leq |E(J[U])| - 2(m+1) + 2m < |E(J[U])|$; this contradicts our choice of U. Therefore $p_i = 2(m+1)$.

The second claim follows from the fact that if $|U| < 2s(m+1)$ and $G[U]$ has the minimum number of edges then we can add $2s(m+1) - |U|$ vertices of degree at most $2m$. The third claim is proved by similar arguments. □

Reduction: Now we describe our reduction.

1. A copy of a gadget $F(k)$ is constructed. Denote this graph by F_R and let $V(F_R) = \{x^R_{i1}, x^R_{i2} | 1 \leq i \leq k+m+1\}$.
2. For each $i \in \{1, \ldots, n\}$, a copy of a gadget $F(c(u_i))$ is created. Denote this graph by F_{u_i} and let $V(F_{u_i}) = \{x^{u_i}_{j1}, x^{u_i}_{j2} | 1 \leq j \leq c(u_i)+m+1\}$.
3. For each $i \in \{1, \ldots, r\}$, a copy of a gadget $F(1)$ is created. Denote this graph by F_{v_i} and let $V(F_{v_i}) = \{x^{v_i}_{j1}, x^{v_i}_{j2} | 1 \leq j \leq m+2\}$.
4. For each $e \in E(G)$, the vertex w_e is constructed.
5. For each $i \in \{1, \ldots, n\}$, let $\{e_1, \ldots, e_{d_i}\} = E(u_i)$ for $d_i = d_G(u_i)$. We consider the vertices $w_{e_1}, \ldots, w_{e_{d_i}}$; these vertices are joined by edges to the vertices x^R_{i1}, x^R_{i2} of F_R, and for each $j \in \{1, \ldots, d_i\}$, w_{e_j} is joined by edges to the vertices $x^{u_i}_{j1}, x^{u_i}_{j2}$ of F_{u_i}.
6. For each $i \in \{1, \ldots, r\}$, let $\{e_1, \ldots, e_{d_i}\} = E(v_i)$ for $d_i = d_G(v_i)$. We consider the vertices $w_{e_1}, \ldots, w_{e_{d_i}}$ and for each $j \in \{1, \ldots, d_i\}$, w_{e_j} is joined by edges to the vertices $x^{v_i}_{j1}, x^{v_i}_{j2}$ of F_{v_i}.
7. Create $2m+1$ vertices $z_1, \ldots, z_{2m+1}$ and join them to all vertices w_e for $e \in E(G)$.

Denote the obtained graph by H (see Figure 1).



Graph $F(l)$ for $l = 2$ and $m = 2$

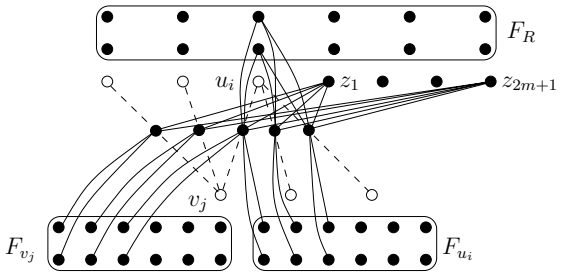


Figure 1: Construction of H.

The proof of correctness of the reduction is based on the next two lemmas.

Lemma 2. *The red-blue graph G has a capacitated dominating set of size at most k if and only if H contains an induced subgraph with $2(m+1)(n+r+1)+2m+1+r$ vertices and at most $2m(m+1)(n+r+1)+r(2m+1)$ edges.*

Proof. Suppose that the red-blue graph G has a capacitated dominating set S of size at most k. Let f be a corresponding domination mapping. We construct an induced subgraph Q of H as follows:

- Include all the vertices $z_1, \ldots, z_{2m+1}$ in the set of vertices of Q.
- For F_R, select $m+1$ sets of the $(k+m+1)$-partition $\{x^R_{i1}, x^R_{i2}\}$ such that $u_i \notin S$ and include these vertices in $V(Q)$.
- For each $u_i \in S$, consider $c \leq c(u_i) \leq d_G(u_i)$ edges $e_1, \ldots, e_c \in E(u_i)$ which are used for domination (i.e. for each $e_j = u_i v_h$, $f(v_h) = u_i$). Select $m+1$ sets of the $(c(u_i)+m+1)$-partition $\{x^{u_i}_{j1}, x^{u_i}_{j2}\}$ of F_{u_i} such that $x^{u_i}_{j1}, x^{u_i}_{j2}$ are not adjacent to $w_{e_1}, \ldots, w_{e_c}$ and include these vertices in $V(Q)$.
- For each $u_i \in R \setminus S$, select $m+1$ sets of the $(c(u_i)+m+1)$-partition $\{x^{u_i}_{j1}, x^{u_i}_{j2}\}$ of F_{u_i} arbitrarily and include these vertices in $V(Q)$.
- For each $v_i \in B$, let $e = v_i f(v_i) \in E(G)$. Select $m+1$ sets of the $(m+2)$-partition $\{x^{v_i}_{j1}, x^{v_i}_{j2}\}$ of F_{v_i} such that $x^{v_i}_{j1}, x^{v_i}_{j2}$ are not adjacent to w_e and include these vertices in $V(Q)$.
- For each $v_i \in B$, include the vertex w_e for $e = v_i f(v_i) \in E(G)$ in Q.

It is straightforward to check that Q has $2m+1+2(m+1)(n+r+1)+r$ vertices and $2m(m+1)(n+r+1)+r(2m+1)$ edges.

Assume now that Q is an induced subgraph of H with $2(m+1)(n+r+1)+2m+1+r$ vertices and the minimum number of edges, such that $|E(Q)| \leq 2m(m+1)(n+r+1)+r(2m+1)$.

Claim 1. *It can be assumed that $z_1, \ldots, z_{2m+1} \in V(Q)$.*

Proof of Claim 1. Let $Z = \{z_1, \ldots, z_{2m+1}\} \cap V(Q)$ and suppose that $|Z| < 2m+1$. Let $W = \{w_e | e \in E(G)\} \cap V(Q)$. Assume that $W \neq \emptyset$. Suppose that $|W| \geq |Z|$. Recall that m is the number of edges of G and, hence, $|W| \leq m$. Then $|Z| \leq m$ and $|\{z_1, \ldots, z_{2m+1}\} \setminus Z| \geq m+1$. Now we observe that the graph obtained from Q by the replacement of the vertices of W by $|W|$ vertices from $\{z_1, \ldots, z_{2m+1}\} \setminus Z$ contains less edges than Q and this contradicts our choice of Q. If $|W| < |Z|$ then the graph obtained from Q by the replacement of a vertex $u \in W$ by a vertex $v \in \{z_1, \ldots, z_{2m+1}\} \setminus Z$ has at most $|E(Q)| - |Z| + |W| - 1 < |E(Q)|$ edges and we again have a contradiction. Therefore $W = \emptyset$ and $\{z_1, \ldots, z_{2m+1}\}$ is an independent set in Q. Then the replacement of any $2m+1-|Z|$ vertices from $V(Q) \setminus Z$ by the vertices of $\{z_1, \ldots, z_{2m+1}\} \setminus Z$ does not increase the number of edges. □

From now on we assume that $z_1, \ldots, z_{2m+1} \in V(Q)$.

Claim 2. *The set $V(Q)$ contains $2(m+1)(n+r+1)$ vertices of the gadgets F_R, $F_{u_1}, \ldots, F_{u_n}$, $F_{v_1}, \ldots, F_{v_r}$ and r vertices from the set $\{w_e | e \in E(G)\}$. Moreover, each of the gadgets F_R, $F_{u_1}, \ldots, F_{u_n}$, $F_{v_1}, \ldots, F_{v_r}$ contains exactly $2(m+1)$ vertices from $m+1$ sets of the partitions of the gadgets and the vertices from $\{w_e | e \in E(G)\} \cap V(Q)$ are not adjacent to vertices of these gadgets in the graph Q.*

Proof of Claim 2. Let p be the number of vertices of Q in the gadgets $F_R, F_{u_1}, \ldots, F_{u_n}$, $F_{v_1}, \ldots, F_{v_r}$ and let q be the number of vertices in $\{w_e | e \in E(G)\}$.

Suppose that $p < 2(m+1)(n+r+1)$. By the second claim of Lemma 1, Q contains at least $2pm - 2(n+r+1)(m+1)m$ edges in the graphs $F_R, F_{u_1}, \ldots, F_{u_n}, F_{v_1}, \ldots, F_{v_r}$. Each of the q vertices w_e is adjacent to the vertices $z_1, \ldots, z_{2m+1}$ in Q. Observe that $p+q = 2(m+1)(n+r+1)+r$. Hence Q contains at least $2m(m+1)(n+r+1)+(2m+1)m+(2(m+1)(n+r+1)-p) > 2m(m+1)(n+r+1) + r(2m+1)$ edges. This contradiction proves that $p \geq 2(m+1)(n+r+1)$.

Suppose now that $p > 2(m+1)(n+r+1)$. We apply the third claim of Lemma 1 and conclude that Q has at least $(p-2(n+r+1)(m+1))2(m+1) + 2(n+r+1)(m+1)m = (r-q)2(m+1)+2(n+r+1)(m+1)m$ edges. Recall that the q vertices w_e in Q are adjacent to the vertices $z_1, \ldots, z_{2m+1}$ in Q. We have that Q contains at least $(r-q)2(m+1) + 2(n+r+1)(m+1)m + q(2m+1) = 2(n+r+1)(m+1)m + r(2m+1) + r - q > 2(n+r+1)(m+1)m + r(2m+1)$ edges. We conclude that $p = 2(m+1)(n+r+1)$ and $q = r$.

Since $p = 2(m+1)(n+r+1)$ and $q = r$, Q has at most $2m(m+1)(n+r+1)$ edges different from $w_e z_j$. By the first claim of Lemma 1, Q has at least $2m(m+1)(n+r+1)$ edges in the gadgets $F_R, F_{u_1}, \ldots, F_{u_n}$, $F_{v_1}, \ldots, F_{v_r}$. Hence the vertices from $\{w_e | e \in E(G)\} \cap V(Q)$ are not adjacent to the vertices of these gadgets in the graph Q. Also by the same claim, each of the gadgets $F_R, F_{u_1}, \ldots, F_{u_n}, F_{v_1}, \ldots, F_{v_r}$ contains exactly $2(m+1)$ vertices of the $m+1$ sets of the partitions of the graph. □

Now we can complete the proof of the lemma.

The graph Q has r vertices from the set $\{w_e | e \in E(G)\}$. Consider a gadget F_{v_i} constructed for the vertex $v_i \in B$. The graph Q contains $2(m+1)$ vertices in F_{v_i} which constitute $m+1$ sets in the $(m+2)$-partition of this graph. Hence only the unique vertex w_e, for $e \in E_G(v_i)$, non-adjacent to the vertices of F_{v_i} in Q can be included in $V(Q)$. It follows that for each $i \in \{1, \ldots, r\}$, exactly one vertex w_e is included in $V(Q)$. We define the mapping $f\colon B \to R$ by setting $f(v_i) = u_j$ such that $w_e \in V(Q)$ for $e = v_i u_j$. Now observe that $2(m+1)$ vertices of the graph F_R in $m+1$ sets of the $(k+m+1)$-partition are in $V(Q)$. Thus, exactly k sets of vertices from $\{w_e | e \in E_G(u_1)\}, \ldots, \{w_e | e \in E_G(u_n)\}$ are non-adjacent to all the vertices of F_R in Q. We construct the set $S \subset R$ by including in it vertices $u_i \in R$ for which $\{w_e | e \in E_G(u_i)\}$ has this property. Clearly, $|S| = k$. Each $w_e \in V(Q)$ is included in one of the sets $\{w_e | e \in E_G(u_i)\}$ for $u_i \in S$.

It remains to observe that each set $\{w_e | e \in E_G(u_i)\}$ can only contain at most $c(u_i)$ vertices, namely those that are non-adjacent to the $2(m+1)$ vertices of Q in F_{u_i}. We conclude that S is a capacitated dominating set in the red-blue graph G and f is a dominating mapping for S. □

We prove an upper bound for the clique-width of H as a linear function in the clique-width of the incidence graph $I(G)$ of G.

Lemma 3. *We have* $\mathbf{cwd}(H) \leq 9 \cdot \mathbf{cwd}(I(G)) + 1$ *and an expression tree of width at most* $9 \cdot \mathbf{cwd}(I(G)) + 1$ *for* H *can be constructed in polynomial time given an expression tree of width* $\mathbf{cwd}(I(G))$ *for* $I(G)$.

Proof. Let $w = \mathbf{cwd}(I(G))$ and suppose that the expression tree for $I(G)$ uses labels $\{\mu_1, \ldots, \mu_w\}$. To construct an expression tree for H we use the following labels:

- labels $\alpha_1, \ldots, \alpha_w$ and $\alpha'_1, \ldots, \alpha'_w$ for the vertices of F_R;
- labels $\beta_1, \ldots, \beta_w$ for the vertices w_e;
- labels $\gamma_1, \ldots, \gamma_w$, $\gamma'_1, \ldots, \gamma'_w$ and $\delta_1, \ldots, \delta_w$ for the vertices of the gadgets F_{u_i};
- labels $\zeta_1, \ldots, \zeta_w$, $\zeta'_1, \ldots, \zeta'_w$ and $\eta_1, \ldots, \eta_w$ for the vertices of the gadgets F_{v_i}; and
- a working label λ.

We construct the required expression tree for H by going over the expression tree for $I(G)$ and making necessary changes to it.

When a vertex $u_i \in R$ labeled by μ_p is introduced, we perform the following set of operations. We first introduce the vertices x^R_{i1}, x^R_{i2} labeled α_p. Then we construct the subgraph of F_{u_i} induced by the vertices $x^{u_i}_{j1}, x^{u_i}_{j2}$ for $d_G(u_i) < j \leq c(u_i) + m + 1$ by repeating $c(u_i) + m + 1 - d_G(u_i)$ times the following: (a) introduce two vertices labeled λ; (b) join the vertices labeled by λ to the vertices labeled by δ_p; (c) relabel the vertices labeled by λ by δ_p.

When a vertex $v_i \in B$ labeled by μ_p is introduced, we construct the subgraph of F_{v_i} induced by the vertices $x^{u_i}_{j1}, x^{u_i}_{j2}$ for $d_G(v_i) < j \leq m + 2$ by repeating $m+2-d_G(v_i)$ times the following: (a) introduce two vertices labeled λ; (b) join the vertices labeled by λ to the vertices labeled by η_p; (c) relabel the vertices labeled by λ by η_p.

When a vertex of $I(G)$ which corresponds to an edge $e = u_i v_j \in E(G)$ labeled μ_p is introduced, we (a) introduce the vertex w_e labeled by β_p; (b) introduce two vertices of F_{u_i} labeled by γ_p; (c) introduce two vertices of F_{v_j} labeled ζ_p; (d) join the vertex labeled by β_p to the vertices labeled γ_p, ζ_p.

Notice that we omitted the union operations from these descriptions. For each union operation in the expression tree for $I(G)$, we do as follows. If both graphs contain vertices labeled α, then (a) vertices labeled α_p in one of the graphs are relabeled α'_p for $p \in \{1, \ldots, w\}$; (b) we perform the union operation;

(c) the vertices labeled α_p and α'_q are joined for $p, q \in \{1, \ldots, w\}$; and (d) the vertices labeled α'_p are relabeled α_p for $p \in \{1, \ldots, w\}$.

If both graphs contain vertices of the same gadget F_{u_i} labeled γ, then we again modify the tree by replacing the union operation by recursively repeating the following for $i = 1, \ldots, n$: if both graphs contain vertices of F_{u_i} labeled γ, then (a) vertices labeled γ_p in one of the graphs are relabeled γ'_p for $p \in \{1, \ldots, w\}$ if γ_p is a label of a vertex of F_{u_i}; (b) we perform the union operation; (c) the vertices labeled γ_p and γ'_q are joined for $p, q \in \{1, \ldots, w\}$ if γ_p, γ'_q are labels of vertices of F_{u_i}; and (d) the vertices labeled γ'_p are relabeled γ_p for $p \in \{1, \ldots, w\}$.

Similarly, if both graphs contain vertices of the same gadget F_{v_i} labeled ζ, then we proceed by replacing the union operation by recursively repeating the following for $i = 1, \ldots, r$: if both graphs contain vertices of F_{v_r} labeled ζ, then (a) vertices labeled ζ_p in one of the graphs are relabeled ζ'_p for $p \in \{1, \ldots, w\}$ if ζ_p is a label of a vertex of F_{v_i}; (b) we perform the union operation; (c) the vertices labeled ζ_p and ζ'_q are joined for $p, q \in \{1, \ldots, w\}$ if ζ_p, ζ'_q are labels of vertices of F_{v_i}; and (d) the vertices labeled ζ'_p are relabeled ζ_p for $p \in \{1, \ldots, w\}$.

In all other cases we just do the union operation.

If, in the expression tree of $I(G)$, we have a join operation between two labels, say μ_p and μ_q, then we do the following. The vertices labeled α_p and β_q are joined and symmetrically we join the vertices labeled α_q and β_p. For $i = 1, \ldots, n$, if there are vertices of F_{u_i} labeled γ_p, δ_q (γ_q, δ_p, respectively) then (a) the vertices labeled γ_p, δ_q (γ_q, δ_p, respectively) are joined; (b) the vertices labeled γ_p (γ_q, respectively) are relabeled by δ_p (δ_q, respectively). For $i = 1, \ldots, r$, if there are vertices of F_{v_i} labeled ζ_p, η_q (ζ_q, η_p, respectively) then (a) the vertices labeled ζ_p, η_q (ζ_q, η_p, respectively) are joined; (b) the vertices labeled ζ_p (ζ_q, respectively) are relabeled by η_p (η_q, respectively).

The relabel operation in the expression tree of $l(G)$, that is, relabel μ_p to μ_q is replaced by the following relabelings: (a) α_p to α_q; (b) β_p to β_q; (c) γ_p to γ_q; (d) δ_p to δ_q; (e) ζ_p to ζ_q; (f) η_p to η_q.

After we have completed the scanning of the expression tree for $I(G)$, we have to complete the construction of F_R. It remains to create the vertices x^R_{i1}, x^R_{i2} for $n < i \leq k + m + 1$ and join them to other sets of the $(k + m + 1)$-partition. We repeat $k + m + 1 - n$ times the following: (a) introduce two vertices labeled λ; (b) join the vertices labeled λ to the vertices labeled $\alpha_1, \ldots, \alpha_w$; (c) relabel the vertices labeled λ by α_1. □

To complete the proof of Theorem 1, notice that the number of vertices of H and the parameter k are polynomial in $n+r$. Therefore, $\log k$ is linear in $\log(n+k)$, and if we could solve Sparse k-Subgraph in time $2^{o(\mathbf{cwd}(H) \log k)} \cdot |V(H)|^{O(1)}$ then Red-Blue CDS could be solved in time $2^{o(\mathbf{cwd}(I(G)) \log |V(G)|))} \cdot |V(G)|^{O(1)}$. By Corollary 1, it cannot be done unless ETH fails.

4. Regular subgraph problem

In this section we consider the d-Regular Subgraph problem. The main aim is the proof of the following theorem.

Theorem 2. *The d-Regular Induced Subgraph problem can be solved on n-vertex graphs of clique-width at most w in time $d^{O(w)} \cdot n$ if an expression tree of width w is given for the input graph, but it cannot be solved in time $2^{o(w \log d)} \cdot n^{O(1)}$ unless ETH fails, even if an expression tree of width w is given.*

The proof is given in the following two subsections.

4.1. Algorithmic upper bounds for d-Regular Induced Subgraph

We sketch a dynamic programming algorithm for solving d-Regular Induced Subgraph in time $d^{O(w)} \cdot n$ on graphs of clique-width at most w.

Let G be a graph with n vertices, and let T be an expression tree for G of width w. Recall that for a node X of T, we denote by G_X the w-graph associated with this node. For a node X of T, let $U_1(X), \ldots, U_w(X)$ be the sets of vertices of G_X labeled $1, \ldots, w$, respectively. We are looking for an induced d-regular subgraph H of X. For a node X of T, denote by H_X the subgraph of G_X induced by $V(H) \cap V(G_X)$ (it can happen that $V(H) \cap V(G_X) = \emptyset$). We use the following observation.

If X is an introduce node, then H_X has only vertices of degree 0 (or no vertices). If X is a relabel node, then the degrees of vertices of H_X are the same as in H_Y for the unique child Y of X. If X is a union node, then the degree of H_X are the degrees of the corresponding vertices in H_Y and H_Z where Y and Z are the children of X. Finally, if X is a join node $\eta_{i,j}$, then $d_{H_X}(v) = d_{H_Y}(v)+s_j$ if $v \in U_i(X)$ and $d_{H_X}(v) = d_{H_Y}(v)+s_i$ if $v \in U_j(X)$ where Y is the unique child of X and s_i, s_j are the number of vertices of H_X labeled i and j respectively in G_X. All other vertices have the same degrees as in G_Y. In particular, it implies the following.

Observation 1. *Let H be an induced d-regular subgraph of G. Then for any node X of the expression tree T, all vertices of H_X from the set $U_i(X)$ have the same degree (in H) for $i \in \{1, \ldots, w\}$.*

It indicates that each partial solution for a node X can be encoded by the degrees of vertices in each $U_i(X)$ of an induced subgraph of G_X that has exactly s_i vertices of $U_i(X)$ for $i \in \{1, \ldots, w\}$.

Now we describe formally what we store in the tables corresponding to the nodes in an expression tree. Let G be a graph with n vertices and let T be an expression tree for G of width w. Recall that for a node X of T, we denote by G_X the w-graph associated with this node. We let $U_1(X), \ldots, U_w(X)$ be the sets of vertices of G_X labeled $1, \ldots, w$ respectively. The table of data for the node X stores pairs of integer vectors, $(s_1, \ldots, s_w)$ and $(d_1, \ldots, d_w)$, satisfying $0 \leq s_i \leq d+1$ and $0 \leq d_i \leq d$ for all $i \in \{1, \ldots, w\}$, and for which there is an induced subgraph H of G_X such that for all $i \in \{1, \ldots, w\}$

- $d_i = d_H(v)$ for all $v \in V(H) \cap U_i(X)$ (if $V(H) \cap U_i(X) = \emptyset$ then it is assumed that this condition holds for all $0 \leq d_i \leq d$), and
- $s_i = \min\{|V(H) \cap U_i(X)|, d+1\}$.

If X is the root node of T (that is, $G = G_X$) then G contains a d-regular induced subgraph if and only if the table for X contains an entry with the vector $(d_1, \ldots, d_w) = (d, \ldots, d)$.

Now we give the details of how we construct our tables and how we update them.

Introduce Node: Tables for introduce nodes of T are constructed in a straightforward manner. Suppose that $X = i(v)$ for $v \in V(G)$ and $i \in \{1, \ldots, w\}$. Then the table of data for the node X contains the pairs of vectors $(s_1, \ldots, s_w)$ and $(d_1, \ldots, d_w)$ such that $s_j = 0$ and $0 \leq d_j \leq d$ for $j \in \{1, \ldots, w\}$, $j \neq i$, and either $s_i = 0$ and $0 \leq d_i \leq d$ or $s_i = 1$ and $d_i = 0$.

Relabel Node: Suppose that X is a relabel node $\rho_{i \to j}$, and let Y be the child of X. Then the table for X contains a pair of vectors $(s_1, \ldots, s_w)$ and $(d_1, \ldots, d_w)$ if and only if $s_i = 0$, and the table for Y contains the entry $(s'_1, \ldots, s'_w)$, $(d'_1, \ldots, d'_w)$ such that

- $s_p = s'_p$ and $d_p = d'_p$ for $p \in \{1, \ldots, w\}$, $p \neq i, j$,
- $d_j = d'_i = d'_j$,
- $s_j = \min\{s'_i + s'_j, d+1\}$.

Union Node: Let X be a union node with children Y and Z. In this case the table for X contains a pair of vectors $(s_1, \ldots, s_w)$ and $(d_1, \ldots, d_w)$ if and only if the tables for Y and Z have the pairs of vectors $(s'_1, \ldots, s'_w)$, $(d_1, \ldots, d_w)$ and $(s''_1, \ldots, s''_w)$, $(d_1, \ldots, d_w)$, respectively, such that $s_i = \min\{s'_i + s''_i, d+1\}$ for $i \in \{1, \ldots, w\}$.

Join Node: Finally, suppose that X is a join node $\eta_{i,j}$ with the child Y. The table for X has a pair of vectors $(s_1, \ldots, s_w)$ and $(d_1, \ldots, d_w)$ if and only if the table for Y includes a pair of vectors $(s_1, \ldots, s_w)$, $(d'_1, \ldots, d'_w)$ such that

- $d'_p = d_p$ for $p \in \{1, \ldots, w\}$, $p \neq i, j$,
- $d_i = d'_i + s_j$ and $d_j = d'_j + s_i$.

Correctness of the algorithm follows from the description and Observation 1. Since for each X, the table for X contains at most $(d+2)^{2w}$ pairs of vectors, the algorithm runs in time $d^{O(w)} \cdot n$. This proves that d-Regular Induced Subgraph can be solved in time $d^{O(w)} \cdot n$ on graphs of clique-width at most w.

4.2. Algorithmic lower bounds for d-Regular Induced Subgraph

To prove our complexity lower bound, we give a reduction from the Red-Blue Exact Saturated CDS problem, parameterized by the clique-width of the incidence graph of the input graph, to the d-Regular Induced Subgraph problem. The proof is organized as follows: we first give a construction, then prove its correctness and finally bound the clique-width of the transformed instance.

Construction.. Let (G, c, k) be an instance of Red-Blue Exact Saturated CDS with $R = \{u_1, \ldots, u_n\}$ being the set of red vertices and $B = \{v_1, \ldots, v_r\}$ being the set of blue vertices. Let $d = n+r+1$ if $n+r$ is even and let $d = n+r+2$ otherwise; notice that d is odd. We need an auxiliary gadget.

Auxiliary gadget $F(x)$***:*** Let x be a vertex. We construct $\frac{d-1}{2}$ copies of K_{d+1}, subdivide one edge of each copy, and glue (identify) all these vertices of degree two into one vertex y. Finally we join x and y by an edge. We are going to attach gadgets $F(x)$ to other parts of our construction through the vertex x. This vertex is called the *root* of $F(x)$. The properties of $F(x)$ that are required for our proof are summarized in the following simple lemma. The gadget $F(x)$ for $d = 5$ is illustrated in Figure 2.

Lemma 4.

- *Let G' be the graph obtained from a graph G with a vertex x by adding a copy of $F(x)$. For any d-regular induced subgraph H of G', if any vertex of $F(x) - x$ is included in H then $V(F(x)) \subseteq V(H)$ and if $x \notin V(H)$ then $V(F(x)) \cap V(H) = \emptyset$.*
- *We have $\mathbf{cwd}(F(x)) \leq 4$, and a 4-graph isomorphic to $F(x)$ can be constructed in such a way that one label is used only for the vertex x. Moreover it can be assumed that the construction starts from the vertex x with a given label α, and at the end all non-root vertices are relabeled by a given label λ.*

Proof. The first claim of the lemma follows immediately from the fact that all vertices of $F(x)$ except x have degree d.

To prove the second claim, let us note that $F(x)$ is a cograph (i.e. a graph without induced paths on four vertices). It is well-known that cographs have clique-width 2 (see e.g. [36]). It means that in fact $\mathbf{cwd}(F(x)) = 2$. Clearly, by using two additional labels α and λ, we can get the desired 4-graph. □

Reduction: Now we describe our reduction. Recall that $d = n + r + 1$ if $n + r$ is even and $d = n + r + 2$ otherwise. Let $s = d - r - 1$ and $t = d - k - 1$.

1. Vertices $u_1, \ldots, u_n$ are created.

2. A clique of size r with vertices $v_1, \ldots, v_r$ is constructed.

3. For each edge $e = u_iv_j$ of G, a vertex w_e is added, joined by edges to u_i and v_j, and $d-2$ copies of $F(w_e)$ are constructed.

4. A clique of size s with vertices $a_1, \ldots, a_s$ is created, all vertices a_i are joined to vertices $v_1, \ldots, v_r$, and for each $i \in \{1, \ldots, s\}$, a copy of $F(a_i)$ is added.

5. A vertex x is introduced and joined by edges to $v_1, \ldots, v_r$ and $a_1, \ldots, a_s$.

6. A vertex y is added and joined by an edge to x, and $k-1$ copies of $F(y)$ are added.

7. A clique of size t with vertices $b_1, \ldots, b_t$ is constructed, the vertex y is joined by edges to all vertices of the clique, and for each $j \in \{1, \ldots, t\}$, k copies of $F(b_i)$ are added.

8. A vertex z is introduced and joined by edges to vertices y and $b_1, \ldots, b_t$.

9. For each $i \in \{1, \ldots, n\}$, we let $l_i = d - c(u_i) - 1$ and do the following:

- Add a vertex p_i, join it to z by an edge, and construct $c(u_i)-1$ copies of $F(p_i)$.
- Construct a clique of size l_i with vertices $c_{i1}, \ldots, c_{il_i}$, join them to the vertex p_i by edges, and for each $j \in \{1, \ldots, l_i\}$, introduce $c(u_i)$ copies of $F(c_{ij})$.
- Join the vertex u_i to the vertices p_i and $c_{i1}, \ldots, c_{il_i}$ by edges.

Denote the obtained graph by H. The construction of H is illustrated in Figure 2.

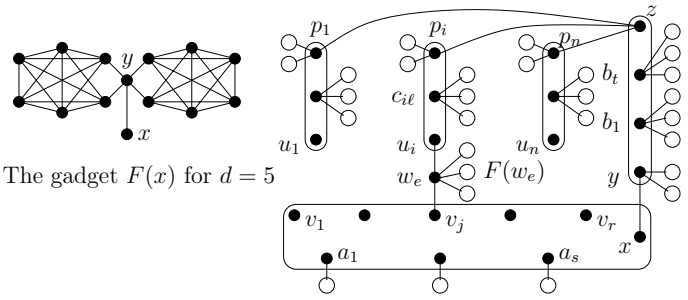


Figure 2: Construction of H.

Lemma 5. *The red-blue graph G has an exact saturated capacitated dominating set of size k if and only if H contains an induced d-regular subgraph.*

Proof. Let S be an exact saturated capacitated dominating set of size k in G and let f be a corresponding domination mapping. We construct an induced d-regular subgraph Q of H as follows:

- All the vertices $v_1, \ldots, v_r$, $a_1, \ldots, a_s$, x, y, $b_1, \ldots, b_t$ and z are included in the set of vertices of Q.
- For each $u_i \in S$, all the vertices u_i, $c_{i1}, \ldots, c_{il_i}$ and p_i are included in the set of vertices of Q.
- For each $i \in \{1, \ldots, r\}$, the vertex w_e for $e = v_i f(v_i) \in E(G)$ is included in the set of vertices of Q.
- For each gadget $F(w)$ rooted at a vertex $w \in V(H)$ such that $w \in V(Q)$, all the vertices of $F(w)$ are included in Q.

It is straightforward to check that the subgraph of H induced by the vertices of Q is a d-regular graph.

Assume now that Q is a d-regular induced subgraph of H. First we need some properties of Q.

Claim 3. *All the vertices $v_1, \ldots, v_r, a_1, \ldots, a_s, x, y, b_1, \ldots, b_t, z \in V(Q)$.*

Proof of Claim 3. Assume for the sake of contradiction that at least one vertex of the set $\{v_1, \ldots, v_r, a_1, \ldots, a_s, x\}$ is not in $V(Q)$. Then all the vertices $a_1, \ldots, a_s, x, y$ are not in $V(Q)$ since they have degree d in H. Each vertex v_i is adjacent to at most n vertices w_e and hence $d_H(v_i) \leq n+r+s \leq d-1+s$. Hence if $a_1, \ldots, a_s \notin V(Q)$ then $v_1, \ldots, v_r \notin V(Q)$. Now we can conclude that all the vertices w_e cannot be vertices of Q and therefore $u_1, \ldots, u_n \notin V(Q)$. Since $y \notin V(Q)$, by the same arguments, $b_1, \ldots, b_t, z \notin V(Q)$, and if $u_1, \ldots, u_n \notin V(Q)$ then all vertices p_i and c_{ij} also cannot be included in $V(Q)$. But this means that $V(Q) = \emptyset$. We easily get the same conclusion if we assume that at least one vertex of the set $\{y, z, b_1, \ldots, b_t\}$ is not in $V(Q)$. □

Claim 4. *There is a k-element set $I \subseteq \{1, \ldots, n\}$ such that for any $i \in I$, we have $u_i, c_{i1}, \ldots, c_{il_i}, p_i \in V(Q)$, and for any $i \notin I$, we have $u_i, c_{i1}, \ldots, c_{il_i}, p_i \notin V(Q)$.*

Proof of Claim 4. By Claim 3, we have $y, b_1, \ldots, b_t, z \in V(Q)$. Hence z should be adjacent in Q to at least k other vertices, and therefore we have that exactly k vertices $p_{i_1}, \ldots, p_{i_k} \in V(Q)$. Let $I = \{i_1, \ldots, i_k\}$. It remains to observe that if $p_j \in V(Q)$ then $p_j, c_{j1}, \ldots, c_{jl_j}, u_j \in V(Q)$, and $p_j, c_{j1}, \ldots, c_{jl_j}, u_j \notin V(Q)$ otherwise. □

By Claim 3, we have $v_1, \ldots, v_r, a_1, \ldots, a_s, x \in V(Q)$. It follows that each vertex v_i is adjacent to exactly one vertex w_e in Q for some edge $e = v_i u_j \in E(G)$, and $u_j \in V(Q)$ since $d_H(w_e) = d$. By Claim 4, we have that exactly k vertices $u_{i_1}, \ldots, u_{i_k} \in V(Q)$. Also by this claim, $u_{i_j}, c_{i_j 1}, \ldots, c_{i_j l_i}, p_{i_j} \in V(Q)$ and hence each vertex u_{i_j} should be adjacent to exactly $c(u_{i_j})$ vertices w_e. We let $S = \{u_{i_1}, \ldots, u_{i_k}\} \subseteq R$ in the red-blue graph G and define a mapping $f\colon B \to R$ by setting $f(v_i) = u_j$. It remains to observe that S is an exact saturated capacitated dominating set and f is a domination mapping for this set. □

Now we show that the clique-width of H is bounded from above by a linear function in the clique-width of the incidence graph $I(G)$ of G.

Lemma 6. *We have that* $\mathbf{cwd}(H) \leq 3 \cdot \mathbf{cwd}(I(G)) + 6$ *and an expression tree of width at most* $3 \cdot \mathbf{cwd}(I(G)) + 6$ *for* H *can be constructed in polynomial time assuming we are given an expression tree of width* $\mathbf{cwd}(I(G))$ *for* $I(G)$.

Proof. Let $w = \mathbf{cwd}(I(G))$ and suppose that the expression tree for $I(G)$ uses labels $\{\alpha_1, \ldots, \alpha_w\}$. To construct an expression tree for H we use the following additional labels:

- labels $\beta_1, \ldots, \beta_w$ and $\gamma_1, \ldots, \gamma_w$ for the vertices $v_1, \ldots, v_r$;
- a label δ for vertices p_i;
- working labels ζ_1, ζ_2 for the vertices $a_1, \ldots, a_s$, $b_1, \ldots, b_t$, $c_{i1}, \ldots, c_{il_i}$, x, y and z;
- working labels η_1, η_2 for non-root vertices of gadgets $F(w)$; and
- a label λ for vertices of $F(w)$ and for vertices that are not joined to other vertices in further stages of the construction.

We construct the required expression tree for H by going over the expression tree for $I(G)$ and making necessary changes to it.

When a vertex $u_i \in R$ labeled by α_p is introduced, we perform the following set of operations. We first introduce the vertex u_i labeled α_p. Then l_i vertices $c_{i1}, \ldots, c_{il_i}$ are created by repeating the following operations l_i times: (a) introduce a vertex labeled ζ_1; (b) join vertices labeled ζ_1 to the vertices labeled α_p and ζ_2; (c) construct copies of $F(c_{ij})$ rooted at this vertex using the labels η_1, η_2 as it is shown in Lemma 4 (recall that when a copy of $F(u_i)$ is constructed here and further, all the non-root vertices are relabeled by λ); (d) relabel vertices labeled ζ_1 by ζ_2. Finally, (a) the vertex p_i labeled by δ_1 is introduced; (b) the vertex is joined to the vertices labeled α_p and ζ_2; (c) copies of $F(p_i)$ rooted at this vertex are constructed; (d) all the vertices labeled ζ_2 are relabeled λ. We omit the union operations from our descriptions here and henceforth in any similar descriptions and assume that if some vertex is introduced then union is always performed.

When a vertex of $I(G)$ that corresponds to an edge $e \in E(G)$ labeled α_p is introduced, we introduce the vertex w_e also labeled α_p and add copies of $F(w_e)$ rooted at this vertex.

When a vertex $v_i \in B$ labeled by α_p is introduced, we introduce the vertex v_i labeled β_p.

For each union operation in the expression tree for $I(G)$, we do as follows. If both graphs contain vertices labeled β, then (a) vertices labeled β_p in one of the graphs are relabeled γ_p for $p \in \{1, \ldots, w\}$; (b) we perform the union operation; (c) the vertices labeled β_p and γ_q are joined for $p, q \in \{1, \ldots, w\}$; and (d) the vertices labeled γ_p are relabeled β_p for $p \in \{1, \ldots, w\}$. If only one graph contains vertices labeled β then we just do the union operation.

If, in the expression tree of $I(G)$, we have a join operation between two labels, say α_p and α_q, then we simulate this by applying the join operations between the following: (a) α_p and α_q; (b) α_p and β_q; (c) β_p and α_q.

A relabel operation in the expression tree of $I(G)$, say an operation relabeling α_p to α_q, is replaced by the following relabeling process: (a) α_p to α_q and (b) β_p to β_q.

After we have completed the scanning of the expression tree for $I(G)$, we complete the construction of H. We create vertices $a_1, \ldots, a_s$ by repeating the following operations s times: (a) introduce a vertex labeled ζ_1; (b) join vertices labeled ζ_1 to the vertices labeled by labels β_p for $p \in \{1, \ldots, w\}$ and ζ_2; (c) construct copies of $F(a_i)$ rooted at this vertex using the labels η_1, η_2; (d) relabel vertices labeled ζ_1 by ζ_2. Then (a) the vertex x labeled by ζ_1 is introduced; (b) it is joined to all the vertices labeled ζ_2 and β_p for $p \in \{1, \ldots, w\}$; (c) all the vertices labeled ζ_2 are relabeled by λ. Now (a) the vertex y labeled by ζ_2 is introduced; (b) copies of $F(y)$ rooted at this vertex are added; (c) y is joined to the vertex x labeled ζ_1; (c) all the vertices labeled ζ_1 are relabeled by λ. Then we create vertices $b_1, \ldots, b_t$ by repeating the following operations t times: (a) introduce a vertex labeled ζ_1; (b) join vertices labeled ζ_1 to vertices labeled ζ_2; (c) construct copies of $F(b_i)$ rooted at this vertex; (d) relabel vertices labeled ζ_1 by ζ_2. Finally we (a) introduce the vertex z labeled ζ_1 and (b) join it to all the vertices labeled by ζ_2 or δ_1. □

To conclude this part of the proof of Theorem 2, we observe that the number of vertices of H and the parameter d are polynomial in $n + r$, and therefore if we could solve d-Regular Induced Subgraph in time $2^{o(\mathbf{cwd}(H) \log d)} \cdot |V(H)|^{O(1)}$ then the Red-Blue Exact Saturated CDS could be solved in time $2^{o(\mathbf{cwd}(I(G)) \log |V(G)|))} \cdot |V(G)|^{O(1)}$. By Corollary 1, this cannot be done unless ETH fails.

4.3. Variants of the d-Regular Induced Subgraph problem

We conclude this section by giving a corollary for some related problems. In the d-Regular Induced Subgraph problem we ask about the existence of a d-regular induced subgraph for a given graph. It is possible to get similar results for some variants of this problem. The Minimum d-Regular Induced Subgraph problem and the Maximum d-Regular Induced Subgraph problem are respectively the problems of finding a d-regular induced subgraph of minimum and maximum size. For the Counting d-Regular Induced Subgraph problem, we are interested in the number of induced d-regular subgraphs of the input graph. Using Theorem 2 we get the following corollary.

Corollary 3. *The* Minimum d-Regular Induced Subgraph, Maximum d-Regular Induced Subgraph *and* Counting d-Regular Induced Subgraph *problems can be solved on n-vertex graphs of clique-width at most w in time $d^{O(w)} \cdot n$ if an expression tree of width w is given, but they cannot be solved in time $2^{o(w \log d)} \cdot n^{O(1)}$ unless ETH fails, even if an expression tree of width w is given.*

Proof. To prove the algorithmic upper bounds, it is sufficient to modify the algorithm described for d-Regular Induced Subgraph. Particularly for Minimum d-Regular Induced Subgraph, each entry of the table of data for the node X should store a positive integer p and pairs of vectors $(s_1,\ldots,s_w)$ and $(d_1,\ldots,d_w)$ of integers such that $0 \leq s_i \leq d+1$ and $0 \leq d_i \leq d$ for $i \in \{1,\ldots,w\}$, for which p is the number of vertices of an induced subgraph H of G_X of minimum size such that for $i \in \{1,\ldots,w\}$

- $d_i = d_H(v)$ for all $v \in V(H) \cap U_i(X)$ (observe that if $V(H) \cap U_i(X) = \emptyset$ then it is assumed that this condition holds for any $0 \leq d_i \leq d$), and
- $s_i = \min\{|V(H) \cap U_i(X)|, d+1\}$.

For Maximum d-Regular Induced Subgraph and Counting d-Regular Induced Subgraph, the parameter p should be the number of vertices of an induced subgraph of maximum size and the number of subgraphs, respectively.

The lower bounds follow from Theorem 2 immediately. □

5. Minimum Subgraph of Minimum Degree at least d

In this section we consider the Minimum Subgraph of Minimum Degree at least d problem.

Theorem 3. *The* Minimum Subgraph of Minimum Degree at least d *problem can be solved on n-vertex graphs of clique-width at most w in time $d^{O(w)} \cdot n$ if an expression tree of width w is given, but it cannot be solved in time $2^{o(w \log d)} \cdot n^{O(1)}$ unless ETH fails, even if an expression tree of width w is given.*

Again we first prove algorithmic upper bounds and then establish lower bounds.

5.1. Algorithmic upper bounds for Minimum Subgraph of Minimum Degree at least d

The Minimum Subgraph of Minimum Degree at least d problem can be solved by a variant of the dynamic programming algorithm for d-Regular Induced Subgraph.

Let G be a graph with n vertices and let T be an expression tree for G of width w. Let $U_1(X),\ldots,U_w(X)$ be the sets of vertices of G_X labeled $1,\ldots,w$, respectively. For an induced subgraph H of G and a node X of T, denote by H_X the subgraph of G_X induced by $V(H) \cap V(G_X)$ (it can happen that $V(H) \cap V(G_X) = \emptyset$). Recall that if X is an introduce node, then H_X has only vertices of degree 0 (or no vertices). If X is a relabel node, then the degrees of vertices of H_X are the same as in H_Y for the unique child Y of X. If X is a union node, then the degree of H_X are the degrees of the corresponding vertices in H_Y and H_Z where Y and Z are the children of X. Finally, if X is a join node $\eta_{i,j}$, then $d_{H_X}(v) = d_{H_Y}(v) + s_j$ if $v \in U_i(X)$ and $d_{H_X}(v) = d_{H_Y}(v) + s_i$ if $v \in U_j(X)$ where Y is the unique child of X and s_i, s_j are the number of

vertices of H_X labeled i and j respectively in G_X. All other vertices have the same degrees as in G_Y. It shows that each partial solution for a node X can be encoded by the minimum degrees of vertices in each $U_i(X)$ of an induced subgraph of G_X that has exactly s_i vertices of $U_i(X)$ for $i \in \{1, \ldots, w\}$.

We describe formally what we store in the tables corresponding to the nodes in the expression tree. Let G be a graph with n vertices and let T be an expression tree for G of width w. Let $U_1(X), \ldots, U_w(X)$ be the sets of vertices of G_X labeled $1, \ldots, w$, respectively. Each entry of the table of data for the node X stores a positive integer p and pairs of vectors $(s_1, \ldots, s_w)$ and $(d_1, \ldots, d_w)$ of integers such that $0 \leq s_i \leq d$ and $0 \leq d_i \leq d$ for $i \in \{1, \ldots, w\}$, for which p is the number of vertices of an induced subgraph H of G_X of minimum size such that for $i \in \{1, \ldots, w\}$

- $d_i = \min\{d, \min\{d_H(v) | v \in V(H) \cap U_i(X)\}\}$ (it is assumed that $d_i = d$ if $V(H) \cap U_i(X) = \emptyset$), and
- $s_i = \min\{|V(H) \cap U_i(X)|, d\}$.

If X is the root node of T (that is, $G = G_X$) then G contains an induced subgraph of degree at least d with at most k vertices if and only if the table for X contains an entry with the parameter $p \leq k$ and vector $(d_1, \ldots, d_w)$ such that $d_i \geq d$ for $i \in \{1, \ldots, w\}$.

Now we give the details of how we construct our tables and how we update them.

Introduce Node: Tables for introduce nodes of T are constructed in a straightforward manner. Suppose that $X = i(v)$ for $v \in V(G)$ and $i \in \{1, \ldots, w\}$. Then the table of data for the node X contains the entries for $p = 0$ and $p = 1$. For $p = 0$, the table stores the pairs of vectors $(s_1, \ldots, s_w)$ and $(d_1, \ldots, d_w)$ such that $s_j = d$ and $d_i = d$ for $j \in \{1, \ldots, w\}$. For $p = 1$, the table contains the pairs of vectors $(s_1, \ldots, s_w)$ and $(d_1, \ldots, d_w)$ such that $s_j = d$ and $d_i = d$ for $j \in \{1, \ldots, w\}$, $j \neq i$, and $s_i = 1$, $d_i = 0$.

Relabel Node: Suppose that X is a relabel node $\rho_{i \to j}$, and let Y be the child of X. Then the table for X contains an entry with an integer p and a pair of vectors $(s_1, \ldots, s_w)$ and $(d_1, \ldots, d_w)$ if and only if $s_i = 0$ and p is the minimum integer for which the table for Y contains the entry with p and the vectors $(s'_1, \ldots, s'_w)$, $(d'_1, \ldots, d'_w)$ such that

- $s'_q = s_q$ and $d'_q = d_q$ for $q \in \{1, \ldots, w\}$, $q \neq i, j$,
- $d_i = \min\{d'_i, d'_j\}$,
- $s_j = \min\{s'_i + s'_j, d\}$.

Union Node: Let X be a union node with children Y and Z. In this case the table for X contains an entry with p and a pair of vectors $(s_1, \ldots, s_w)$ and $(d_1, \ldots, d_w)$ if and only if p is the minimum integer for which the tables for Y and Z have the entries p', $(s'_1, \ldots, s'_w)$, $(d'_1, \ldots, d'_w)$ and p'', $(s''_1, \ldots, s''_w)$, $(d''_1, \ldots, d''_w)$, respectively, such that

- $p = p' + p''$,
- $d_i = \min\{d'_i, d''_i\}$ for $i \in \{1, \ldots, w\}$, and
- $s_i = \min\{s'_i + s''_i, d\}$ for $i \in \{1, \ldots, w\}$.

Join Node: Finally, suppose that X is a join node $\eta_{i,j}$ with the child Y. It can be noted that the table for X has an entry p, $(s_1, \ldots, s_w)$, $(d_1, \ldots, d_w)$ if and only if p is the minimum integer for which the table for Y includes the entry p, $(s_1, \ldots, s_w)$, $(d'_1, \ldots, d'_w)$ such that

- $d'_q = d_q$ for $q \in \{1, \ldots, w\}$, $q \neq i, j$,
- $d_i = \min\{d'_i + s_j, d\}$ and $d_j = \min\{d'_j + s_i, d\}$.

Correctness of the algorithm follows from the description of the procedure.

Since for each X, the table for X contains at most $(d+1)^{2w}$ pairs of vectors and for each pair of vectors only one value of the parameter p is stored, the algorithm runs in time $d^{O(w)} \cdot n$. This proves that Minimum Subgraph of Minimum Degree at least d can be solved in time $d^{O(w)} \cdot n$ on graphs of clique-width at most w.

5.2. Algorithmic lower bounds for Minimum Subgraph of Minimum Degree at least d

To prove our lower bounds we give a reduction from the Red-Blue CDS problem, parameterized by the clique-width of the incidence graph of the input graph, to the Minimum Subgraph of Minimum Degree at least d problem. The proof is based on the same ideas as the proof for the d-Regular Induced Subgraph problem.

Again we first give a construction, then prove its correctness, and finally prove that the clique-width of the graph in the reduced instance is bounded by a linear function in the clique-width of the incidence graph of the original graph.

Construction.. Let (G, c, k) be an instance of Red-Blue CDS with $R = \{u_1, \ldots, u_n\}$ being the set of red vertices and $B = \{v_1, \ldots, v_r\}$ being the set of blue vertices. Let $d = n + r + 2$ if $n + r$ is odd and let $d = n + r + 3$ otherwise; notice that d is odd.

We need some auxiliary gadgets. Here we use the gadget $F(x)$ defined in the proof of Theorem 2. Observe that $F(x)$ has $f = \frac{(d+1)(d-1)}{2} + 1$ non-root vertices of $F(x)$. Now we construct graphs $Q_i(u, x, D)$ for $i \in \{1, \ldots, n\}$.

Auxiliary gadget $Q_i(u, x, D)$***:***

1. For each $j \in \{1, \ldots, n\}$, let $l_j = d - c(u_j) - 2$ and do the following:

- Introduce two adjacent vertices p_j, q_j, and construct $c(u_j) - 1$ copies of $F(p_j)$ and $F(q_j)$.
- Introduce a vertex g_j and join it to p_j and q_j.

– Construct a clique of size l_j with vertices $c_{j1}, \ldots, c_{jl_j}$, join them to the vertices g_j, p_j and q_j by edges, and for each $h \in \{1, \ldots, l_j\}$, introduce $c(u_j)$ copies of $F(c_{jh})$.

2. For each $j \in \{1, \ldots, n\}$, $j \neq i$, do the following:

– Introduce vertices $d_{j1}, \ldots, d_{jc(u_j)}$ and join them to g_j.
– Construct $d-2$ copies of $F(d_{jh})$ for $h \in \{1, \ldots, c(u_j)\}$.

3. For $j \in \{1, \ldots, n-1\}$, the edges $q_j p_{j+1}$ are introduced.

4. A vertex x is introduced, $d-3$ copies of $F(x)$ are constructed, and finally x is joined by edges to p_1 and q_n.

Now let $u = g_i$ and let $D = \{d_{jh} | 1 \leq j \leq n, j \neq i, 1 \leq h \leq l_j\}$. We attach gadgets $Q_i(u, x, D)$ to other parts of our construction by the vertices u, x and join D to other vertices in such a way that all the vertices of D are adjacent to one vertex. The construction of $Q_i(u, x, D)$ is illustrated in Figure 3. The properties of $Q_i(u, x, D)$ are given in the following lemma.

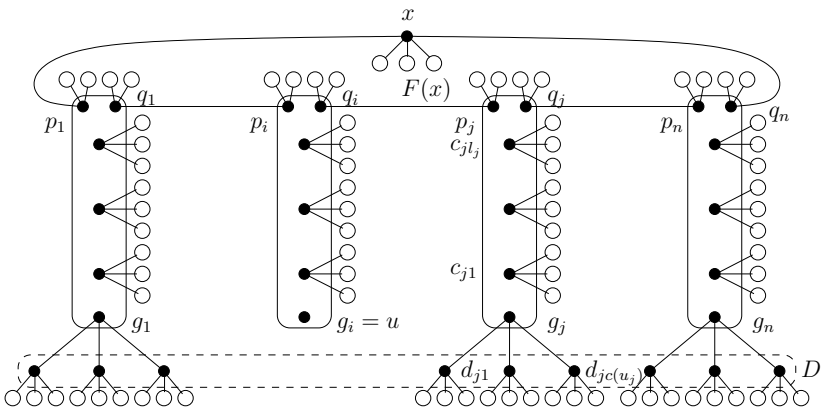


Figure 3: Construction of $Q_i(u, x, D)$.

Lemma 7.

- *Let G' be the graph obtained from a graph G with vertices u, x and w by adding a copy of $Q_i(u, x, D)$ and joining w to all the vertices of D by edges. For any induced subgraph H of G' of minimum degree at least d, if any vertex of $Q_i(u, x, D) - \{u, x\}$ is included in H then $V(Q_i(u, x, D)) \subseteq V(H)$ and if $u \notin V(H)$ or $x \notin V(H)$ then $(V(Q_i(u, x, D)) \setminus \{u, x\}) \cap V(H) = \emptyset$.*
- *We have that $\mathbf{cwd}(Q_i(u, x, D)) \leq 10$, and the 10-graph isomorphic with $Q_i(u, x, D)$ can be constructed in such a way that one given label α is used only for the vertex u, another given label β is used only for x, all the vertices of D are relabeled by a given label γ, and all other vertices are relabeled by a given label λ at the end of the construction.*

- $|V(Q_i(u,x,D))| = q - c(u_i)(d-2)f$ *where* $q = \sum_{j=1}^{n}[(2dc(u_j) - c(u_j)^2 - 2c(u_j) - 2)f + 1]$.

Proof. The first claim of the lemma follows immediately from the fact that all the non-root vertices of $Q_i(u,x,D)$ have degree d.

To prove the second claim, we sketch how to construct a 10-graph isomorphic to $Q_i(u,x,D)$ using the following labels:

- labels α, β for the vertices u and x, respectively;
- labels γ and η for the vertices d_{jh};
- labels δ_1, δ_2 for the vertices p_j, q_j, c_{jh} and q_j $(j \neq i)$;
- labels ζ_1, ζ_2 for the non-root vertices of the gadgets $F(w)$; and
- label λ for vertices that are not joined to other vertices in the further stages of the construction.

First, we introduce the vertex x labeled α.

For $j = 1, \ldots, i-1$ we do the following. We introduce the vertex p_j labeled by δ_1. If $j > 1$ then we join it to the vertex q_{j-1} labeled δ_2 and relabel the vertices labeled δ_2 by λ. If $j = 1$ then the vertex x labeled α is joined to the vertex p_j. Then $c(u_j) - 1$ copies of $F(p_j)$ rooted at p_j are created using the labels ζ_1, ζ_2 as described in Lemma 4 (recall that when a copy of $F(w)$ is constructed, all the non-root vertices are relabeled by λ). After that the vertex p_j is relabeled by δ_2. Next, we repeat the following operations l_j times. (a) introduce the vertex c_{jh} labeled δ_1; (b) join vertices labeled δ_1 to the vertices δ_2; (c) construct copies of $F(c_{jh})$ rooted at this vertex using the labels ζ_1, ζ_2; (c) relabel vertices labeled δ_1 by δ_2. Next, the vertex g_j labeled by δ_1 is added and joined to vertices labeled δ_2. Then we deal with the vertices d_{jh} by repeating the following $c(u_j)$ times: (a) create a vertex labeled η; (b) join it to g_j labeled δ_1; (c) create $d-2$ copies of $F(d_{jh})$ rooted at this vertex; (d) relabel the vertices labeled by η by γ. Finally, we (a) relabel g_j by δ_2; (b) introduce q_j labeled δ_1; (c) join it to all the vertices labeled δ_2; (d) construct $c(u_j) - 1$ copies of $F(q_j)$; (e) relabel all the vertices labeled δ_2 by λ; (f) relabel q_j by δ_2.

For $j = i$, we use almost the same operations. The only differences are that we label $u = g_i$ by β, we do not relabel it, we do not repeat the operations described above for d_{jh}, and finally we join q_i to the vertex labeled β. For $j = i+1, \ldots, n$, our construction is the same as for $j \in \{1, \ldots, i-1\}$. The construction is concluded by joining x labeled by α to q_n labeled by γ_2 and relabeling q_n by λ.

The third claim of the lemma follows immediately from the description of $Q_i(u,x,D)$. □

Reduction: Now we describe our reduction. Let $s = d-r-2$ and $t = d-k-1$.

1. Vertices $u_1, \ldots, u_n$, $h_1, \ldots, h_n$ are introduced and for each $i \in \{1, \ldots, n\}$ a copy of a gadget $Q_i(u_i, h_i, D_i)$ is created (u_i, h_i are the roots, and by D_i we denote the set of vertices D for this copy of a gadget).

2. A clique of size r with vertices $v_1, \ldots, v_r$ is created.

3. For each edge $e = u_i v_j$ of G, a vertex w_e is added, joined by edges to u_i and v_j, and $d-2$ copies of $F(w_e)$ are constructed.

4. A clique of size s with vertices $a_1, \ldots, a_s$ is created, all vertices a_i are joined to vertices $v_1, \ldots, v_r$, and for each $i \in \{1, \ldots, s\}$, a copy of $F(a_i)$ is added.

5. A vertex x is introduced and joined by edges to $v_1, \ldots, v_r$ and $a_1, \ldots, a_s$.

6. A vertex w is introduced and joined to $v_1, \ldots, v_r$, $a_1, \ldots, a_s$, x and all the vertices of the sets $D_1, \ldots, D_n$.

7. A vertex y is introduced and joined by an edge to x, and $k-1$ copies of $F(y)$ are added.

8. A clique of size t with vertices $b_1, \ldots, b_t$ is created, the vertex y is joined by edges to all the vertices of the clique, and for each $j \in \{1, \ldots, t\}$, k copies of $F(b_i)$ are added.

9. A vertex z is introduced and joined by edges to the vertices y, $b_1, \ldots, b_t$ and all the vertices $h_1, \ldots, h_n$.

Denote the obtained graph by H (see Figure 4).

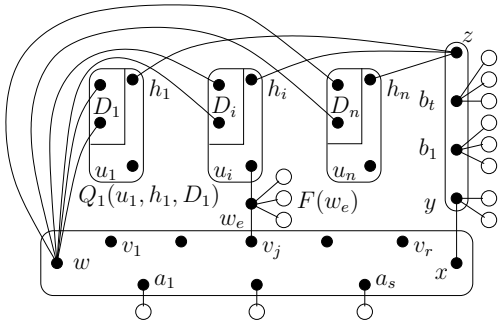


Figure 4: Construction of H.

Lemma 8. *The red-blue graph G has a capacitated dominating set of size at most k if and only if H contains an induced subgraph with minimum degree at least d which has at most $kq + n + s(f+1) + (k-1)f + t(kf+1) + 4$ vertices.*

Proof. Suppose that the red-blue graph G has a capacitated dominating set of size at most k. Clearly, we can assume that there is a capacitated dominating set S of size exactly k. Let f be a corresponding domination mapping. We construct an induced subgraph J of H as follows:

- Include all the vertices $v_1, \ldots, v_r$, $a_1, \ldots, a_s$, x, w, y, $b_1, \ldots, b_t$ and z in J.

- For each $u_i \in S$, all the vertices of $Q_i(u_i, h_i, D_i)$ are included in J.
- For each $i \in \{1, \ldots, r\}$, the vertex w_e for $e = v_i f(v_i) \in E(G)$ is included in the set of vertices of J. For each $j \in \{1, \ldots, n\}$, if $|f^{-1}(u_j)| < c(u_j)$ then $c(u_j) - |f^{-1}(u_j)| < c(u_j)$ additional vertices w_e which correspond to the edges incident with u_j in G are arbitrarily chosen and included in J.
- For each gadget $F(p)$ rooted at a vertex $p \in V(J)$, all the non-root vertices of $F(p)$ are included in J.

It is straightforward to check that J has $kq+n+s(f+1)+(k-1)f+t(kf+1)+4$ vertices and $d_J(p) \geq d$ for $p \in V(J)$.

Assume now that J is an induced subgraph of H with at most $kq+n+s(f+1)+(k-1)f+t(kf+1)+4$ vertices such that $d_J(p) \geq d$ for $p \in V(J)$. We need some properties of J.

Claim 5. *All the vertices $v_1, \ldots, v_r, a_1, \ldots, a_s, x, w, y, b_1, \ldots, b_t, z \in V(J)$.*

Proof of Claim 5. Assume for the sake of contradiction that at least one vertex of the set $\{v_1, \ldots, v_r, a_1, \ldots, a_s, x, w\}$ is not in $V(J)$. Then all the vertices $a_1, \ldots, a_s, x, y \notin V(J)$ since they have degree d in H. Each vertex v_i is adjacent to at most n vertices w_e and hence $d_H(v_i) \leq n+r+s+1 \leq d-1+s$. Hence if $a_1, \ldots, a_s \notin V(J)$ then $v_1, \ldots, v_r \notin V(J)$. Now we see that all the vertices w_e cannot be vertices of J and therefore $u_1, \ldots, u_n \notin V(J)$. By the first claim of Lemma 7, all the vertices of the gadgets $Q(u_i, h_i, D_i)$ are not included in J and also $w \notin V(J)$. Since $y \notin V(J)$, by similar arguments $b_1, \ldots, b_t, z \notin V(J)$. This means that $V(J) = \emptyset$. We easily get the same conclusion if we assume that at least one vertex of the set $\{y, z, b_1, \ldots, b_t\}$ is not in J. □

Claim 6. *There is a k-element set $I \subseteq \{1, \ldots, n\}$ such that for any $i \in I$, $V(Q_i(u_i, h_i, D_i)) \subseteq V(J)$, and for any $i \notin I$, none of the vertices of $Q_i(u_i, h_i, D_i)$ is included in J.*

Proof of Claim 6. By Claim 5, $y, b_1, \ldots, b_t, z \in V(J)$. Hence z is adjacent in J to at least k other vertices, and therefore there are $k' \geq k$ vertices $h_{i_1}, \ldots, h_{i_{k'}}$ in $V(J)$. By the first claim of Lemma 7, all the vertices of $Q_{i_j}(u_{i_j}, h_{i_j}, D_{i_j})$ for $j \in \{1, \ldots, k'\}$ are in J. Now we observe that for each $j \in \{1, \ldots, k'\}$, at least $c(u_{i_j})$ vertices w_e adjacent to u_{i_j} should be included in J. Using the third claim of Lemma 7, we conclude that J has at least $k'q+n+s(f+1)+(k-1)f+t(kf+1)+4$ vertices. So, $k' = k$ and we let $I = \{i_1, \ldots, i_{k'}\}$. □

Now we complete the proof of the lemma. By Claim 5, we have that $v_1, \ldots, v_r, a_1, \ldots, a_s, x \in V(J)$. It follows that each vertex v_i is adjacent to at least one vertex w_e in J for some edge $e = v_i u_j \in E(G)$, and $u_j \in V(J)$ since $d_H(w_e) = d$. We define a mapping $f\colon B \to R$ by setting $f(v_i) = u_j$. By Claim 6, exactly k vertices $u_{i_1}, \ldots, u_{i_k} \in V(J)$. We let $S = \{u_{i_1}, \ldots, u_{i_k}\} \subseteq R$ in the red-blue graph G. Also by this claim, all the vertices of $Q_{i_1}(u_{i_1}, h_{i_1}, D_{i_1}), \ldots, Q_{i_k}(u_{i_k}, h_{i_k}, D_{i_k})$ are included in $V(J)$, and

hence each vertex u_{i_j} should be adjacent to at least $c(u_{i_j})$ vertices w_e. Observe that if there is a vertex u_{i_j} which is adjacent to more than $c(u_{i_j})$ vertices w_e, then $|V(J)| > kq+n+s(f+1)+(k-1)f+t(kf+1)+4$. Therefore each vertex u_{i_j} is adjacent to exactly $c(u_{i_j})$ vertices w_e. This means that S is a capacitated dominating set and f is a domination mapping for this set. □

Our next aim is to give an upper bound for the clique-width of H as a linear function in the clique-width of the incidence graph $I(G)$ of G.

Lemma 9. *We have* $\mathbf{cwd}(H) \leq 3 \cdot \mathbf{cwd}(I(G)) + 12$*, and an expression tree of width at most* $3 \cdot \mathbf{cwd}(I(G)) + 12$ *for* H *can be constructed in polynomial time, given an expression tree of width* $\mathbf{cwd}(I(G))$ *for* $I(G)$*.*

Proof. Let $w = \mathbf{cwd}(I(G))$ and suppose that the expression tree for $I(G)$ uses labels $\{\alpha_1, \ldots, \alpha_w\}$. To construct an expression tree for H we use the following additional labels:

- labels $\beta_1, \ldots, \beta_w$ and $\gamma_1, \ldots, \gamma_w$ for the vertices $v_1, \ldots, v_r$;
- labels δ_1, δ_2 for the vertices h_i and for the vertices in the sets D_i;
- working labels ζ_1, ζ_2 for the vertices $a_1, \ldots, a_s$, $b_1, \ldots, b_t$, $c_{i1}, \ldots, c_{il_i}$, x, y, z and w;
- working labels $\eta_1, \ldots, \eta_7$ for the non-root vertices of the gadgets $F(w)$ and $Q_i(u_i, h_i, D_i)$; and
- a label λ for vertices of $F(w)$ and for vertices that are not joined to other vertices in further stages of the construction.

We construct the required expression tree for H by going over the expression tree for $I(G)$ and making necessary changes to it.

When a vertex $u_i \in R$ labeled by α_p is introduced, we create $Q_i(u_i, h_i, D_i)$ as described in Lemma 7 using the label α_p for u_i, the label δ_1 for h_i and δ_2 for the vertices of D_i; the labels $\eta_1, \ldots, \eta_7$ are used for the construction of $Q_i(u_i, h_i, D_i)$ and λ is used for the final relabeling.

When a vertex of $I(G)$ which corresponds to an edge $e \in E(G)$ labeled α_p is introduced, we introduce the vertex w_e (labeled α_p) and add $d-2$ copies of $F(w_e)$ rooted at this vertex using labels η_1, η_2 and λ.

When a vertex $v_i \in B$ labeled by α_p is introduced, we introduce the vertex v_i labeled β_p.

For each union operation in the expression tree for $I(G)$, we do the following. If both graphs contain vertices labeled β, then (a) vertices labeled β_p in one of the graphs are relabeled γ_p for $p \in \{1, \ldots, w\}$; (b) we perform the union operation; (c) the vertices labeled β_p and γ_q are joined for $p, q \in \{1, \ldots, w\}$; and (d) the vertices labeled γ_p are relabeled β_p for $p \in \{1, \ldots, w\}$. If only one graph contains vertices labeled β then we just do the union operation.

If in the expression tree of $I(G)$, we have join operation between two labels, say α_p and α_q, then we simulate this by applying join operations between the following: (a) α_p and α_q; (b) α_p and β_q; (c) β_p and α_q.

A relabel operation in the expression tree of $I(G)$, say relabeling α_p to α_q, is replaced by the following relabeling process: (a) α_p to α_q and (b) β_p to β_q.

After we have completed the scanning of the expression tree for $I(G)$, we perform the following operations to complete the construction of H. We create vertices $a_1, \ldots, a_s$ by repeating the following operations s times: (a) introduce a vertex labeled ζ_1; (b) join vertices labeled ζ_1 to the vertices labeled β_p for $p \in \{1, \ldots, w\}$ and ζ_2; (c) construct copies of $F(a_i)$ rooted at this vertex using the labels η_1, η_2; (d) relabel vertices labeled ζ_1 by ζ_2. Then (a) the vertex w labeled by ζ_1 is introduced; (b) it is joined to all the vertices labeled ζ_2, δ_2 and β_p for $p \in \{1, \ldots, w\}$; (c) it is relabeled by ζ_2. Now (a) the vertex x labeled by ζ_1 is introduced; (b) it is joined to all the vertices labeled ζ_2 and β_p for $p \in \{1, \ldots, w\}$; (c) all the vertices labeled ζ_2 are relabeled by λ.

Now (a) the vertex y labeled by ζ_2 is introduced; (b) the copies of $F(y)$ rooted at this vertex are added; (c) the vertex is joined to the vertex x labeled ζ_1; (c) all the vertices labeled ζ_1 are relabeled by λ. Then we create vertices $b_1, \ldots, b_t$ by repeating the following operations t times: (a) introduce a vertex labeled ζ_1; (b) join vertices labeled ζ_1 to the vertices labeled ζ_2; (c) construct copies of $F(b_i)$ rooted at this vertex; (d) relabel vertices labeled ζ_1 by ζ_2. Finally we (a) introduce the vertex z labeled ζ_1 and (b) it is joined to all the vertices labeled by ζ_2 or δ_1. □

To finish the proof of Theorem 2, we observe that the number of vertices of H and the parameters d and k are polynomial in $n + r$, and therefore if we could solve Minimum Subgraph of Minimum Degree at least d in time $2^{o(\mathbf{cwd}(H) \log d)} \cdot |V(H)|^{O(1)}$ then the Red-Blue CDS could be solved in time $2^{o(\mathbf{cwd}(I(G)) \log |V(G)|)} \cdot |V(G)|^{O(1)}$. By Corollary 1, this cannot be done unless ETH fails.

6. Conclusion

We established tight algorithmic lower and upper bounds for some double-parameterized subgraph problems when the clique-width of the input graph is one of the parameters. We believe that similar bounds could be given for other problems.

Another interesting task is to consider problems parameterized by other width-parameters. Throughout the paper, in all our results we assumed that an expression tree of the given width is part of the input. This is crucial, since — unlike the case of tree-width — to date we are unaware of an efficient (FPT or polynomial) algorithm for computing an expression tree with a constant factor approximation of the clique-width. The algorithm given by Oum and Seymour in [32] provides a constant factor approximation for another graph parameter — *rank-width* [24, 32]. Hence, it is natural to ask whether it is possible to

establish tight algorithmic bounds for Dense k-Subgraph, d-Regular Induced Subgraph and Minimum Subgraph of Minimum Degree at least d parameterized by the rank-width of the input graph. Also it would be interesting to consider problems parameterized by the tree-width. For example, it can be shown that d-Regular Induced Subgraph and Minimum Subgraph of Minimum Degree at least d can be solved in time $d^{O(t)} \cdot n$ for n-vertex graphs of tree-width at most t. Is this bound asymptotically tight?

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